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THE PRINCIPLES OF ALTERNATE-CURRENT WORKING

THE PRINCIPLES OF
ALTERNATE-CURRENT WORKING.

BY

ALFRED HAY, B.Sc. (Edin.),

Lecturer on Electrotechnics at the University College, Liverpool.

ILLUSTRATED.

LONDON
BIGGS AND CO., 139 140, SALISBURY COURT, FLEET STREET, E.C.

PREFACE.

This book is the outcome of a course of lectures originally delivered to a class of evening students. It is so extremely elementary in treatment throughout, and the knowledge—both mathematical and electrical—which it pre-supposes on the part of the reader is so very slight, that it is hoped few will find serious difficulty in reading it.

Brief historical notes and references have been appended to each chapter. The reader who intends making a study of the book is strongly recommended to work through all the examples which it contains.

To those with a more advanced mathematical knowledge, the book may serve as a simple and easily digested introduction to standard treatises on the subject, such as Fleming's "Alternate-Current Transformer" and Jackson's "Alternating Currents and Alternating-Current Machinery.'

A. H.

University College,
Liverpool,
October, 1897.

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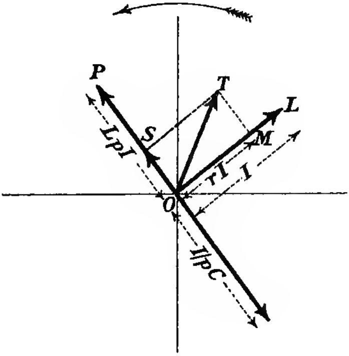
ERRATA.

Page 2, line 3 from foot, for “P P” read “P_2 P.”
„ 40 „ 6 „ top „ “complete” read “complex.”
„ 55 „ 10 „ foot „ “$\frac{}{100}$th” read “$\frac{1}{100}$th.”
„ 75 „ 6 „ top „ “$= 2 B_3$” read “$- 2 B_3$.”
„ 75 „ 8 „ „ „ “$\sqrt{3}\,(e_4 - e_5)$” read “$\sqrt{3}\,(e_1 - e_5)$.”
„ 118 „ 8 „ „ „ “equation 23” read “equation 20.”
„ 119, after exercise 5, add “the periodicity is 100 ∼ per sec.”
„ 156, replace the diagram there given by the following one:



Page 171, exercise 4, second line, for “·4 ampere” read “·51 ampere.”
„ 257, line 2 from top, for “generating” read “receiving.”
„ „ after exercise 2 add “One cubic inch of copper weighs ·32lb.”

THE PRINCIPLES OF ALTERNATE-CURRENT WORKING.

CHAPTER I.

The subject of alternating currents, at one time considered to be one of the most abstruse in the domain of electrical engineering, has recently been brought within the mental reach of many students to whom formerly the task of grappling with it appeared quite hopeless. This is mainly due to the development of special elementary methods for dealing with alternate-current problems, for the introduction of which we are indebted to Blakesley, Kapp, Ferraris, and others. In this we have merely one illustration of the fact that the clearer the insight we gain into natural phenomena, the easier does it become to interpret them by the aid of methods which do not make too large a demand on the mental powers of those whose previous training is somewhat inadequate.

In what follows it will be supposed that the student is familiar with a little elementary algebra, and that he has mastered the fundamental principles of continuous-current phenomena. A knowledge of the C.G.S. electro-magnetic system of units and their connection with the practical units will be assumed. If the student has studied trigonometry, so much the better. But for the sake of those whose mathematical studies have not extended beyond algebra the little trigonometry that will be required in what follows will be briefly explained. Should the explanations given be found too brief, the student is advised to refer to one of the many elementary text-books on the subject.

1. Take two straight lines, X′ X and Y′ Y (Fig. 1), intersecting each other at right angles in the point O.

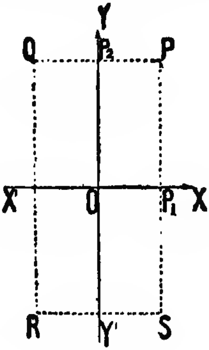


FIG. 1.

The position of any point P in the plane of the paper may be specified by assigning its perpendicular distances $P P_1$ and $P P_2$ from the two lines X′ X and Y′ Y respectively. The two reference lines X′ X and Y′ Y are termed the two *co-ordinate axes.* Their point of intersection O is called the *origin.* The distance P P $(= O P_1)$ of P from the vertical axis is called the *abscissa* of P, and its distance $P_1 P$ $(= O P_2)$ from the horizontal axis is called the *ordinate* of P. The abscissa and ordinate of P are sometimes spoken of as the *co-ordinates* of the point P. If x and y stand for the abscissa and ordinate of P respectively, the point P is denoted by enclosing its co-ordinates in brackets thus: (x, y).

Let P Q R S be a rectangle, such that $P_1 S = P_1 P$, and $P_2 Q = P_2 P$. Then it is obvious that the co-ordinates of all the four points P, Q, R, and S have the same *numerical* values. In order to avoid ambiguity, we agree to regard distances measured to the right of the axis Y′ Y as positive, those measured to the left as negative; distances measured above X′ X as positive, below X′ X as negative. Hence, if x and y denote the *numerical* values of $O P_1$ and $O P_2$ respectively, the four points P, Q, R, S taken in succession are denoted thus:

$$(x, y); \ (-x, y); \ (-x, -y); \ (x, -y),$$

and there is now no ambiguity. The positive direction of the horizontal axis is thus from left to right, and the positive direction of the vertical axis from below upwards, as indicated by the arrow-heads in Fig. 1.

The problem of finding the position of a point P whose co-ordinates x and y are given presents no difficulty. We have merely to lay off from O a distance $O P_1 = x$ along the horizontal axis, and a distance $O P_2 = y$ along the vertical axis. Through the two points P_1 and P_2 we draw two lines parallel to the two axes respectively, and their intersection determines the point required.

2. Let a straight line O A (Fig. 2) be made to turn

round one of its extremities, O, in a *counter-clockwise* direction until it comes into the position O A′. The angle A O A′ $=\theta$ which the line has swept out is reckoned *positive*. If the line had swept out an equal angle A O A″ by turning in a *clockwise* direction round O, the angle would have been reckoned *negative*. Angles are commonly measured in degrees, minutes, and seconds. But for mathematical purposes it is much more convenient to adopt another method of measurement, which is as follows: Measure the length of the arc A A′, and divide it by the length of the radius O A; the quotient gives the *circular* or *radian* measure of the angle A O A′. It is obviously immaterial what radius we choose; for by taking a shorter radius, such as O B, we merely alter the scale of our diagram, but do not change the ratio of the arc (B B′)

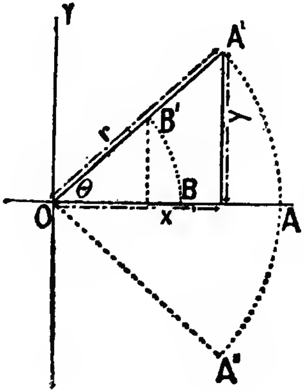


FIG. 2.

to the radius (O B). If the ratio of the arc to the radius is unity, we have a unit angle, or *radian.* Since the length of the circumference of a circle of radius r is $2\pi r$, the radian measure of four right angles is 2π; of two right angles, π; of one right angle, $\frac{\pi}{2}$; and so on.

Let, in Fig. 2, O be taken as origin, and O A as one of the co-ordinate axes. Through O draw the other co-ordinate axis O Y. Let the length of the revolving line O A′ be r, let θ stand for the angle A O A′, and let x, y denote the co-ordinates of the point A′. Then

$\frac{x}{r}$ is called the cosine of θ, and is denoted by $\cos\theta$;

$\frac{y}{r}$ is called the sine of θ, and is denoted by $\sin\theta$;

$\frac{y}{x}$ is called the tangent of θ, and is denoted by $\tan\theta$;

These ratios are called *trigonometrical* or *circular functions* of the angle θ.

An inspection of Fig. 2 will easily convince the student of the truth of the following statements:

The cosine and sine of an angle are capable of assuming any values between -1 and $+1$.

The tangent of an angle may have any positive or negative value, however great numerically. As the angle approaches a right angle $\left(\frac{\pi}{2}\text{ in radian measure}\right)$, its tangent increases indefinitely.

It is hardly necessary to draw the student's attention to the fact that a change in the length of the line O A′ ($=r$) will merely alter the scale of our diagram, and will not affect the values of the ratios $\frac{x}{r}$, $\frac{y}{r}$, and $\frac{y}{x}$.

3. When two quantities are so related that the value of one of them is dependent on that of the other, we say that the first quantity is a *function* of the second. Thus, if we consider the current in a circuit of given resistance, we know that the current will depend on the E.M.F. We may therefore say that the current is a function of the E.M.F. Again, taking the E.M.F. of a given dynamo running at a certain speed with a given external resistance, we may say that the E.M.F. of the dynamo is a function of the speed (*i.e.*, it is dependent on, or determined by, the speed). Corresponding to any value of one of the quantities considered, we have a certain value of the other quantity. This at once suggests a method of graphically representing the connection between the two quantities, for we have merely to regard the values of one of the quantities as the abscissæ of certain points whose ordinates are the corresponding values of the other quantity or function. Such a series of points will lie on a certain curve, and the curve shows us at a glance how the two quantities are related. Such a curve is termed the *graph* of the corresponding function.

4. In most of the cases which will claim our attention we shall have to deal with quantities whose values are constantly varying as time goes on, so that their values will depend on the particular instant of

time at which we consider them; in other words, we shall have to deal with quantities which are *functions of the time.* The variation of such functions is conveniently represented, as explained in the preceding section, on a sheet of squared paper, by plotting the values of the time as abscissæ, and the corresponding values of the function as ordinates. Fig. 3 is an example of how this may be done, and exhibits the changes in the E.M.F. of a certain alternator during ·004 of a second.

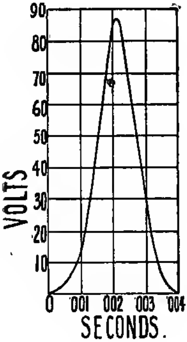


Fig. 3.

Most of the time-functions with which we shall have to deal exhibit a certain peculiarity—viz., they run through a certain cycle of values which is repeated time after time. Such functions are termed *periodic* time-functions. The time taken by the function to go through the complete cycle of values to which it is restricted is called the *periodic time*, or simply the *period*, of the function. If, then, we choose any one value which the function is capable of assuming, this value will be repeated at equal intervals of time, each interval being equal to the period of the function.

5. Let us consider any function of the time, and study the manner in which the function changes as time goes on. In the case of some functions we may find that equal changes in the function take place in equal intervals of time. When this is the case, the function is said to change at a *constant rate,* and we define the *rate of change* of the function as the *change which it undergoes in unit time.* In the case of most functions with which we shall have to deal, however, the changes which take place in the value of the function during equal intervals of time are unequal. In such a case we say that the rate of change of the function is variable, and we define as the *average rate of change* of the function *during any interval of time* the change which takes place in the function, divided by the time during which that change has taken place.

Let P Q R (Fig. 4) represent the graph of a time-function, and let us suppose that as the function changes its graph is recorded by a tracing point. When the point arrives at the position P, it is moving along the tangent to the curve at that point. Let us suppose that during the next unit of time the tracing point *goes on* moving along the tangent instead of following the graph along the curve P Q. If $P_1 Q_1$ represent a unit of time, the point will arrive at the position S at the end of the unit of time. Through P draw a line P T parallel to the axis of time. Then the change in the value of the function which would have taken place in a unit of time had the tracing point followed the tangent at P is represented by T S; we define T S as the *instantaneous rate of change* of the function at P, or its rate of change at the instant when its value is $P_1 P$. Now, T S is the distance through which the tracing point would have moved *in a vertical direction* had it followed the tangent at P—*i.e.*, T S is the vertical velocity of the tracing point when in the position P.

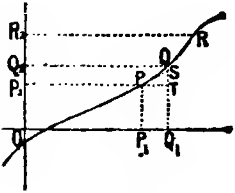


FIG. 4.

We thus arrive at the result that the instantaneous rate of change of a function is given by the vertical velocity of the point which traces out its graph.

6. The *orthogonal projection* of any point P (Fig. 5) on a straight line A B is defined as the foot P_1 of the perpendicular $P P_1$ drawn from P to A B. The orthogonal projection of any curve P Q R on the straight line A B is obtained by dropping perpendiculars $P P_1$ and $R R_1$ from its two extremities P and R on the straight line; $P_1 R_1$ is defined as the orthogonal projection of P Q R on A B. Instead of saying "orthogonal projection," we shall in future use the term "projection" simply. It will be seen

that the projection of any curve on the straight line A B merely depends on the position of the extremities of the curve, and is quite independent of its shape. Thus, in Fig. 5, the projection of the curve P Q R is precisely the same as the projection of the straight line P R joining the extremities of the curve.

Through P draw P S parallel to A B. Then $PS = P_1R_1$ = projection of straight line P R on A B. Also, by definition (Art. 2), $\frac{PS}{PR} = \cos SPR$, so that $P_1R_1 = PS = PR \cos SPR$. We thus have the following proposition: *the projection of any given straight line on another straight line is equal to the length of that line, multiplied by the cosine of its angle of inclination.* By the "angle of inclination" is meant the angle between the two lines.

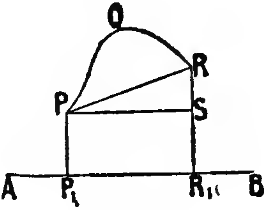


FIG. 5.

7. Returning now to the graph of the function shown in Fig. 4, let us consider the projection on the vertical axis of the point which traces out the graph. As this point passes through the positions P, Q, R, its projection on the vertical axis passes through the positions P_2, Q_2, R_2 respectively. From

the explanation regarding the rate of change of a function given in Art. 5, it will now be evident that this rate of change is given by the *velocity with which the projection of the tracing point moves along the vertical axis.* This result, which is of great importance, we shall have occasion to use later on.

8. There are certain relations connecting the sine, cosine, and tangent ot a given angle which will be required in what follows, and which we shall now briefly establish. Returning to Fig. 2, we see from its geometry that

$$x^2+y^2=r^2,$$

whence

$$\left(\frac{x}{r}\right)^2+\left(\frac{y}{r}\right)^2=1,$$

or,

$$\cos^2\theta + \sin^2\theta=1 \quad . \quad . \quad . \quad . \quad (1)$$

where $\cos^2\theta$ stands for "the square of the cosine of θ," with a similar notation for the sine of θ.

Next,

$$\frac{x}{r}\bigg/\frac{y}{r}=\frac{x}{y},$$

or,

$$\frac{\sin\theta}{\cos\theta}=\tan\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Again, by referring to Fig. 2 and remembering the convention regarding signs (Art. 1), the student will readily see that the following relations hold:

$$\cos(-\theta)=\cos\theta \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\sin(-\theta)=-\sin\theta \quad . \quad . \quad . \quad . \quad (4)$$

Next, from Fig. 2 we see that $\frac{x}{r}=\sin\left(\frac{\pi}{2}-\theta\right)$, so that

$$\cos\theta=\sin\left(\frac{\pi}{2}-\theta\right) \quad . \quad . \quad . \quad . \quad (5)$$

Similarly, $$\sin \theta = \cos \left(\frac{\pi}{2} - \theta\right) \quad . \quad . \quad . \quad (6)$$

We have finally to establish some formulæ regarding the sine and cosine of the sum and difference of two angles. This is most readily done by making use of the results established in Art. 6 regarding orthogonal projection. Let, in Fig. 6, P Q be drawn perpendicular to O Q, while P P_1, Q Q_1 are both perpendicular to the horizontal axis, and P P_2, Q Q_2 perpendicular to the vertical axis. Projecting first on the horizontal axis, we have

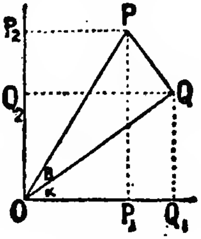


Fig. 6.

projection of O Q = projection of O P + projection of P Q—*i.e.*, bearing in mind (5) and (6),

$$OQ \cos \alpha = OP \cos (\alpha + \beta) + PQ \sin \alpha.$$

Now, since OQ = projection of OP on OQ = OP cos β, and PQ = projection of PO on PQ = PO sin β, the above equation becomes

$$OP \cos \alpha \cos \beta = OP \cos (\alpha + \beta) + OP \sin \alpha \sin \beta,$$

or, $$\cos (\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta \quad . \quad (7)$$

In like manner, by projecting on the vertical axis, we establish the equation

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta \quad . \quad (8)$$

Using now equations (3) and (4), and putting $-\beta$ for β in (7) and (8), we obtain

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta \quad . \quad (9)$$

$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta \quad . \quad (10)$$

All the formulæ given above are of very great importance, and the student is advised to thoroughly master them before going any further.

Notes on Chapter I.

The student unacquainted with trigonometry will find it useful to consult some simple book on this subject, such as Lock's "Trigonometry for Beginners" (Macmillan and Co.). It is of the highest importance that the student should conscientiously work through the exercises given at the end of each chapter, as it is only by this method that a real grasp of the subject can be obtained.

In plotting curves on squared paper, it is a matter of some importance to choose suitable scales for the abscissæ and ordinates; otherwise the actual shape of the curve may be largely disguised. The student will have to exercise his own judgment on this point in plotting the curves of the following examples.

The method explained in Art. 5 for finding the instantaneous rate of change of a function does not, as the student will soon discover for himself, admit of any high degree of accuracy; but for approximate determinations it is frequently useful.

EXERCISES.

1. On a sheet of squared paper draw the graphs of the sine, cosine, and tangent, taking the values of the angle as abscissæ, and those of the corresponding function (sine, cosine, or tangent) as ordinates.

2. A body originally at rest begins to move with a gradually increasing velocity, and its velocity at the end of the 1st, 2nd, 3rd, 4th, and 5th seconds respectively is 5, 14, 27, 44, and 65. Find the

average rate of change of the velocity (*a*) during the 2nd and 3rd; (*b*) during the 4th and 5th seconds. Ans. . 11 and 38.

(3) Draw the graph of the velocity in the preceding example, and determine graphically the *instantaneous* rate of change of the velocity at the end of the 1st, 2nd, 3rd, and 4th seconds. Ans. : 7, 11, 15, and 19.

4. A beaker is filled with water at a temperature of 15deg. C. Heat is then applied to the beaker, and the temperature of the water is found to rise as follows:

Time in Minutes	0	1	2	3	4	5	6	7	8	9	10
Temperature in degrees C.	15	20	24·4	28·4	32	35·2	38·2	41	43·3	45·6	47·7

Plot the time-temperature curve, and from it find: (*a*) the average rate of increase of temperature during the first 6 seconds; (*b*) the instantaneous rates of change at the end of the 2nd, 4th, 6th, and 8th seconds. Ans : 3·87, 4·2, 3·4, 2·8, and 2·36.

CHAPTER II.

9. All physical quantities may be divided into two classes—viz., *scalar* and *vector* quantities. A *scalar* quantity has *magnitude*, but no direction. A *vector* quantity possesses both *magnitude and direction.* Thus, a sum of money, the temperature of the water or steam inside a boiler, the power of an engine, are all examples of *scalar* quantities. The characteristic of a scalar quantity is that it is possible to specify it completely by a simple number. On the other hand, the magnetic force at any point, the velocity of a moving body, the magnetic induction at any point of a mass of iron, are examples of *vector* quantities, and something more than a single number is required to specify them completely. Thus, if we take magnetic force, a single number would merely tell us what the intensity of the force is, without, however, informing us as to the *direction* in which it acts. Vector quantities, then, are quantities having direction in space. Now, direction may be represented by a straight line drawn in space. This at once suggests that we may make use of straight lines to represent vector quantities. The direction of a vector quantity is represented by a straight line with an *arrow-head* affixed to it. Further, the *magnitude* of a vector quantity is easily represented to any convenient scale by the *length* of a line, so

that we have a method of representing vector quantities completely (both as regards magnitude and direction) by straight lines of definite length with arrow-heads attached to them.

Thus, let O A (Fig. 7) be a straight line of length 10, and let us suppose that it represents the magnetic force at the point O. The direction of the force is indicated by the direction of the line O A and its arrow-head. The magnitude of the force is 10, if we suppose each unit of length to correspond to a unit force. The force is therefore completely specified by the straight line O A.

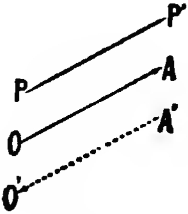


FIG. 7.

Vector quantities are frequently, for the sake of brevity, termed *vectors* simply, and the same term is applied to the lines representing them. In what follows, we shall frequently apply the term "vector" to denote a straight line of definite length, with an arrow-head at one end of it.

10. Hitherto we have regarded vectors (in the sense last named) as representing certain *quantities*. But there is another way of looking at them which is very instructive, and which will enable us easily

to deduce some important results concerning them. We may, in fact, regard a vector as an *operator*, and the operation which we suppose it capable of performing is that of transferring a point from a given position in space to another position, through a distance equal to the length of the vector, and in a direction parallel to it. Thus, the result of applying the vector O A (Fig. 7) to the point P is to transfer the point from P to P′, where P′ is such that P P′ is equal and parallel to O A.

It may be as well at this stage to draw the student's attention to the fact that when a vector is denoted by two letters, such as O A in Fig. 7, the last letter always refers to the extremity of the vector to which the arrow-head is attached. Hence A O would denote a vector parallel and equal in length to O A, but with its arrow-head pointing in the opposite direction, as shown by the dotted line A′ O′ in Fig. 7.

Let us apply the vector O A to the point P; this will transfer P to P′. Next, let the vector A O be applied to P′. This will bring P′ back to its original position P. Thus the successive application of the two vectors O A and A O produces no resultant effect. Hence we may write $\overline{OA} + \overline{AO} = 0$, or $\overline{OA} = -\overline{AO}$, where the bar above the letters is used to indicate that they denote vectors, and not mere lengths.

11. We may now consider the result of applying any two vectors, O A and O B (Fig. 8), to a given point P. Applying first the vector O A, we transfer P to P′, where P P′ = O A; next, applying O B, we transfer P′ to P″, where P′ P″ = O B. The final result, then, is to transfer P to P″. But this might have been

accomplished more simply by at once transferring P to P″ along the straight line P P″. Now $\overline{PP''} = \overline{OC}$, the diagonal of the parallelogram constructed on O A and O B as sides. We thus have $\overline{OA} + \overline{OB} = \overline{OC}$, a result which is generally referred to as the *parallelogram law* for the addition of vectors. Again, we have $\overline{PP'} + \overline{P'P''} = \overline{PP''}$. This result is sometimes spoken of as the *triangle of vectors*.

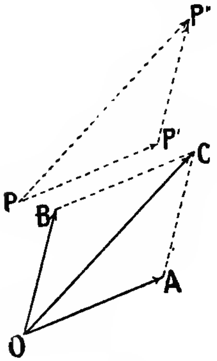


FIG. 8.

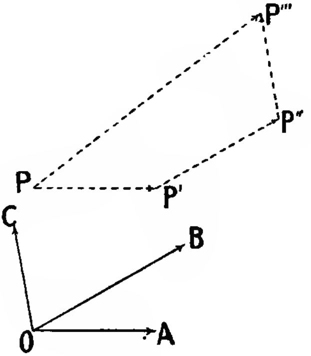


FIG. 9.

It will, of course, at once be seen that the above result only applies to the lines O A, O B, etc., when these are *considered as vectors*, and not as mere lengths, in which latter case it obviously does not hold good.

The law of addition just established for two vectors may be extended to any number of vectors. Thus, let $\overline{OA}$, $\overline{OB}$, and $\overline{OC}$ (Fig. 9) be applied to the point P in succession. The result is to transfer P to P‴ along P P′ P″ P‴. The same result is obtained by the direct application of the single vector P P‴. We thus have $\overline{PP'} + \overline{P'P''} + \overline{P''P'''} = \overline{PP'''}$, a result known as the *polygon of vectors*.

The law established above for the composition of vectors at once provides us with a method for finding the resultant of any number of vector quantities. We thus have the triangle (or parallelogram) and polygon of velocities, accelerations, forces, etc.

12. From what has already been said regarding projection (Art. 6), it will be understood that the projection of a vector on a given straight line is equal to the product of the vector into the cosine of its angle of inclination. Also, the sum of the projections of any two vectors is equal to the projection of their resultant.

13. We now proceed to the study of certain kinds of motion which will be found to be of very great importance in connection with our subject. The two motions which we shall have to consider are known as *uniform circular motion* and *simple harmonic motion* respectively.

Let a point P (Fig. 10), starting from the position A, move with *constant speed* round the circumference of the circle A B C D in a counter-clockwise direction. The motion of P is said to be *uniform circular motion.* As P moves along the circumference of the circle, the radius O P traces out the angle A O P, which increases at a constant rate. The amount by which the angle A O P increases in a unit of time is termed the *angular velocity* of O P. Angular velocity is commonly expressed in *radians per second* (Art. 2). Let us suppose that the angular velocity of O P is p, and let

v stand for the speed of P. Then, since the length of arc described by P in one second is v, and the radius of the circle is equal to O P, the angle swept out by O P in one second is $\frac{v}{\mathrm{O\,P}}$; but this is, by definition, the angular velocity of O P—*i.e.*, p. Hence,

$$p = \frac{v}{\mathrm{O\,P}}, \text{ or, } v = p \,.\, \mathrm{O\,P} \quad . \quad . \quad (11)$$

i.e., the speed of P is obtained by multiplying the angular velocity by the radius of the circle.

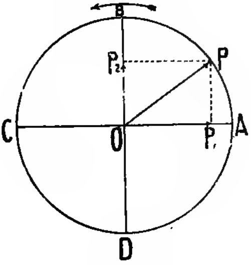


Fig. 10.

14. Consider the projection of P (Fig. 10) on any fixed straight line, such as the horizontal diameter C A of the circle. Let P_1 denote the projection of P on this diameter. As P moves round the circumference of the circle, its projection P_1 *oscillates* to and fro along the diameter C A. This oscillatory motion of P_1 is called a *simple harmonic motion.* We thus define *simple harmonic motion* as the *projection on any fixed straight line of uniform motion in a*

circle. Every simple harmonic motion must therefore be capable of being referred to a corresponding uniform motion in a circle, and this circle is termed the *auxiliary* circle.

The time of a complete revolution in the auxiliary circle is called the *periodic time,* or simply the *period,* of the simple harmonic motion.

The radius of the auxiliary circle is termed the *amplitude* of the simple harmonic motion.

15. The two motions which we have just considered may be represented by means of two vectors as follows. Uniform circular motion may be expressed by a vector of constant length, one of whose extremities is fixed, while the other moves round the circumference of a circle with constant speed. Simple harmonic motion may be represented by another vector, which is the projection, on a fixed straight line, of the first vector. Following Prof. Galileo Ferraris, we shall call the first vector a *rotating vector* and the second an *alternating vector.* A rotating vector is constant in length, but variable in direction; an alternating vector is variable in length, but its direction is always along the same straight line.

Consider two alternating vectors of the same period. At any instant the alternating vectors will be inclined at certain angles to their corresponding rotating vectors. If the two angles are equal we say that the two alternating vectors are in the same *phase*; if the angles are different, the constant difference between them is called the *difference of phase* between the two alternating vectors.

16. We shall now obtain an algebraical expression for an alternating vector. If we suppose that the line in which the alternating vector lies is given, its length at any instant may be expressed by a number, with the *plus* or *minus* sign attached to it. In accordance with the convention regarding signs (Art. 1), a plus sign indicates that the vector has its arrow-head pointing in one direction, a minus sign that it is pointing in the opposite direction. Referring to Fig. 10, let us suppose that a rotating vector O P, of length R, starts from the position O A, and moves round with angular velocity p. Let A O P be the angle swept out in t seconds. Then A O P = $p\,t$. Consider the alternating vector O P_1 obtained by projecting O P on the horizontal diameter. Let r_1 = O P_1. Then

$$\begin{aligned} r_1 &= \mathrm{O\,P_1} = \mathrm{O\,P}\cos \mathrm{A\,O\,P}; \\ &= \mathrm{R}\cos p\,t \quad . \quad . \quad . \quad . \quad (12) \end{aligned}$$

Again, consider the alternating vector O $P_2 = r_2$, obtained by projecting O P on the vertical diameter. We have

$$\begin{aligned} r_2 &= \mathrm{O\,P_2} = \mathrm{O\,P}\cos \mathrm{B\,O\,P}; \\ &= \mathrm{O\,P}\sin \mathrm{A\,O\,P}, \text{ by (5)} \\ &= \mathrm{R}\sin p\,t \quad . \quad . \quad . \quad . \quad (13) \end{aligned}$$

From (12) and (13) we see that an alternating vector may be expressed as the product of two quantities, one of which (R) is a constant, while the other is the *sine or cosine of an angle which varies at a constant rate with the time.*

17. In order to express the *period* T of an alternating vector in terms of the angular velocity p, we have

merely to notice that the angle described by the rotating vector in one period is 2π (*i.e.*, four right angles). Hence $p\,T = 2\pi$, so that $T = \frac{2\pi}{p}$. Also, if n stands for the number of complete periods per second, we have $T = \frac{1}{n}$; hence $p = 2\pi n$.

18. We are now in a position to consider the problem of finding the resultant of two alternating vectors in the same straight line and of the same period. Let O A and O B (Fig. 11) represent the two rotating vectors from which the alternating vectors are derived. Since the two vectors are of the same period, O A and O B will revolve round O with the same angular velocity, so that the angle A O B between them will remain unaltered. This angle is termed the *angle of phase difference* between the two vectors (Art. 15); we may otherwise express this by saying that the alternating vector derived from O A *lags* behind that derived from O B, and that A O B is the *angle of lag*. Let us suppose that the alternating vectors are

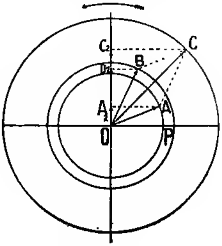


FIG. 11.

obtained by projecting the rotating vectors on the vertical axis. At any instant the *resultant* of the two alternating vectors is the sum of the projections of O A and O B on the vertical axis; but (Art. 12) this is the same as the projection of O C, the diagonal of the parallelogram constructed on O A and O B as sides.

Now since the angle A O B remains constant, O C will remain of constant length, and will revolve about O with the same constant angular velocity as either of the two vectors O A and O B. Hence O C will be a *rotating* vector, and therefore its projection on the vertical axis will be an alternating vector. We have thus established the following proposition:

The resultant of two alternating vectors of the same period and in the same straight line is another alternating vector, also of the same period and in the same straight line.

19. The problem of finding the resultant of two alternating vectors of the same period may always be solved by the graphical method just explained. A reference to Fig. 11 will easily enable the student to see that the resultant vector is always intermediate in phase between its two components, and that the amplitude of the resultant vector is never greater than the sum of the amplitudes of the component vectors, and never less than their difference.

20. The variations of an alternating vector with the time may be represented graphically by plotting a curve whose abscissæ correspond to time, and

whose ordinates represent the corresponding instantaneous values of the alternating vector. This is easily done by the method shown in Fig. 12. Let the circle represent the path of the moving extremity of a rotating vector, and let O D and O Y be the two co-ordinate axes. Divide the circumference into any convenient number of *equal* parts. Since the rotating vector moves with constant angular velocity, each division of the circumference will correspond to the same interval of time. Along the time-axis O D lay off a number of equal lengths, O 1, 12, 23, etc.

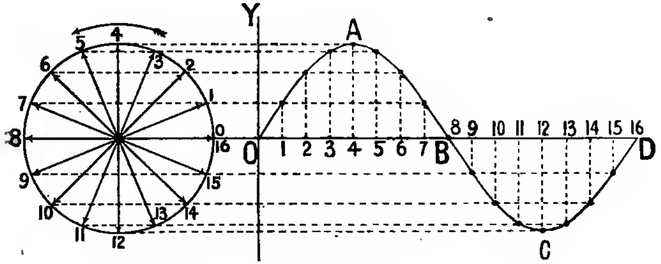


Fig. 12.

Let each of these divisions represent a time-interval equal to that corresponding to one division of the circumference. Through the points of division on the circumference draw lines parallel to the axis of time; the intersections of these lines with the corresponding perpendiculars drawn through the points 1, 2, 3, etc., on the time-axis obviously determine a number of points on the curve required, since the instantaneous values of the alternating vector are determined by projecting the rotating vector on the vertical axis. By drawing a continuous curve through the points so determined we obtain the curve O A B C D, which shows at a glance the way in which the alternating vector varies as time goes on. The time-interval represented by the length O D is obviously equal to the period.

21. Every alternating vector, as we have seen, is capable of being exhibited as the product of two factors, one of which is a certain constant quantity (the *amplitude* of the vector), and the other the sine of the angle which varies at a constant rate as time goes on (Art. 16). Hence the curve O A B C D really represents the variations in the sine of an angle as the angle is varied, the values of the sine being, however, multiplied by the amplitude of the vector. Now, if we multiply each ordinate of a curve by a constant quantity, we merely alter the *scale* of the vertical axis without altering the character of the curve. Hence all such curves as O A B C D are called *sine curves*, or *simple harmonic* curves. One of the characteristics of such curves is that any one of them is capable of being transformed into any other by merely altering the horizontal and vertical scales of the diagram.

22. Any function of the time which is capable of being represented by an alternating vector is termed a *sine* function, or a *simple harmonic* function, of the time. If, for example, we speak of an alternating E.M.F. as being a sine function of the time, we thereby mean that it varies as time goes on according to the law represented by the curve O A B C D in Fig. 12.

In what follows we shall frequently have to deal with the *rate of change of an alternating vector.* In Art. 7 it was explained that if we suppose the graph of a function to be recorded by a tracing point, the rate of change of the function *at any instant* is represented by the velocity of the projection of the tracing point on the vertical axis. Referring to Fig. 13, let O P represent a rotating vector of length A, and angular velocity p. Its projection O P_2 on the vertical axis at any instant gives us an alternating vector, whose rate of change *at the same instant* is represented by the velocity of its extremity P_2. Now the velocity P R of P is, by Art. 13, equal to p A. From O draw a line O S equal and parallel to P R. Then O S represents the velocity of P at the instant considered, and hence the velocity of P_2, the projection of P, will be obtained by projecting O S on the vertical axis. Thus O S_2 gives us the rate of change of the alternating vector at the instant when the latter has the value O P_2, and is increasing. Now

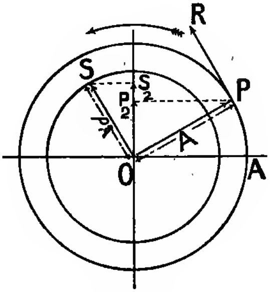


FIG. 13.

since the speed p A of the point P remains constant, O S will be a vector of constant length $= p$ A. Also, since O S is always at right angles to O P, O S will revolve round O with constant angular velocity p—*i.e.*, O S will be a rotating vector, and therefore its projection O S_2 on the vertical axis will be an alternating vector. We thus arrive at the following proposition, which is of very great importance:

The rate of change of an alternating vector of amplitude A and period $\frac{2\pi}{p}$ *(Art.* 17*) is represented by another alternating vector of the same period, and of amplitude p A, this vector being* $\frac{\pi}{2}$ *(one right angle) in advance of the given vector as regards phase.*

Thus, if we have an alternating vector

$$A \sin p\, t,$$

its rate of change is represented by the alternating vector,

$$p\, A \sin \left(p\, t + \frac{\pi}{2}\right).$$

This result may otherwise be expressed thus: the rate of change of a given sine function is represented by another sine function, which is one-quarter of a period in advance of the given function.

In concluding this introductory part of our subject it may be well once more to insist on the extreme importance of the propositions which have just been established. Unless the student has thoroughly mastered them and acquired a clear understanding of what is meant by the *rate of change* of a varying quantity, he cannot hope to make much progress in the study of the principles which underlie alternate-current working.

Notes on Chapter II.

The student may profitably refer to the first two or three chapters of Kelland and Tait's "Introduction to Quaternions" (Macmillan and Co.) for a clear and simple account of the principles dealt with in this chapter. A series of articles by Prof. Galileo Ferraris, published in the *Electrician*, vol. xxxiii., will also be found useful.

The treatment of problems arising in connection with vector quantities has given rise to an important branch of modern mathematics—the *Vector Calculus*. Originally invented by Hamilton, and subsequently developed by Tait, the *Calculus of Quaternions* has recently undergone a number of modifications at the hands of various writers (notably Mr. Oliver Heaviside), and several kinds of *vector algebra* have been devised.

EXERCISES.

1. Draw three vectors, of lengths 10, 15, and 20, the angle between the 1st and 2nd being 20deg. and that between the 2nd and 3rd 30deg., and find the length of the resultant vector, and the angle which it makes with the 1st vector.

2. An armature, 10in. in diameter, is running at 1,000 revolutions per minute. Find (*a*) its angular velocity in radians per second; (*b*) its peripheral speed in feet per second.

3. Two alternating vectors of the same period, and of amplitudes 10 and 8 respectively, are found at a certain instant to have the values + 5 and − 4 respectively, and ·0001 second later their values are + 10 and + 4. Find (*a*) the angle of phase difference; (*b*) the period, assuming this to be greater than ·0001 second.

4. The magnetic flux through a certain circuit varies according to the simple harmonic law, its amplitude being 16,000, and its period ·01 second. Find the greatest rate at which the magnetic flux changes.

CHAPTER III.

23. After the brief exposition given above of the mathematical principles which we shall have to make use of, we may proceed to consider the fundamental physical facts on which the theory of alternating currents is based.

If the number of magnetic lines passing through any circuit is varied, an E.M.F. is induced in that circuit, which is measured, at any instant, by the rate of change in the number of lines at that instant. This is the principle underlying the construction of all kinds of dynamo-electric machines, and was discovered by Faraday as far back as 1831. The student is supposed to be already familiar with this principle, but it may not be out of place to direct his attention to some special points which will prove useful later on.

Lines of magnetic induction may be made to pass through a circuit either by placing it in a suitable position in the neighbourhood of a magnetised body (or conductor conveying a current) or by passing a current round the circuit. Changes in the number of lines which thread themselves through the circuit may be produced by (*a*) moving the circuit in the field in which it is placed; (*b*) moving the body producing the field; (*c*) altering the current which flows round the circuit; (*d*) distorting the shape of the circuit. The first two methods are made use of in the construction of dynamos and alternators, the third in the construction of transformers and various types of alternate-current motors.

The E.M.F. induced in the circuit lasts only so long as the number of magnetic lines passing through it is varying, and ceases as soon as that number becomes constant (when the rate of change, which determines the E M.F., is zero).

24. The *direction* of the induced E.M.F. is such that it always *opposes* the changes producing it. This principle is known as *Lenz's law*. Thus, let the vector A B in Fig. 14 indicate the magnetic field, and let C C_1 represent the circuit. If the length of A B is *diminished* the induced E.M.F. will have the direction indicated by the full-line arrow, and the current due to this E.M.F. would tend to *increase* the length of A B. On the other hand, if we suppose that A B is *lengthened*, the E.M.F. will have the direction shown by the dotted arrow, and the corresponding current would tend to *shorten* A B.

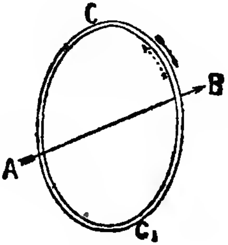


FIG. 14.

Let us next suppose that the line A B gives the direction of the field due to a current flowing in C C_1. The direction of this current will be that of the full-line arrow. If the current is diminished, the E.M.F. induced during the time that the current is diminishing will have the direction shown by the full-line arrow—*i.e.*, the induced E.M.F. will tend to increase the current; if, on the other hand, the current be made to increase, the induced E.M.F. (dotted arrow) will tend to reduce the current. In either case, therefore, *the induced E.M.F. tends to oppose the changes which give rise to it.*

25. The ratio of the E.M.F. to the current in a given circuit maintained at a constant temperature is defined as the resistance of that circuit. Ohm's law asserts that this ratio is a *constant quantity which is independent of the current.* Hence, Ohm's law may be written in the form

$$i = \frac{E}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

where i stands for the current, E for the E.M.F., and r for a certain *constant* which we agree to call the resistance of the circuit.

In applying Ohm's law to any circuit it must be remembered that E in the above equation denotes the total resultant E.M.F. in the circuit; or, if we are dealing with a portion of the circuit merely, E stands for the difference of potential across that portion, plus any E.M.F.'s which may be included in that portion. In many cases there may be more than one E.M.F. in a circuit, and in all such cases it is the *algebraical sum* of the various E.M.F.'s that must be used when applying equation (14). In order to impress this fact on the student's mind, we shall consider a few simple examples.

Let a battery of E.M.F. E, and resistance b, be connected up in simple circuit (Fig. 15) with an external resistance r. Since the E.M.F. of the battery is the only E.M.F. in the circuit, we at once obtain the current i from the formula $i = \frac{E}{b + r}$. Again, P and Q being the positive and negative terminals of the battery, since P is at a higher potential than Q, a current will tend to flow from P to Q. In the branch P R Q we have no E.M.F., so that if V denote the potential difference between P and Q, the current flowing from P to Q along P R Q is equal to $\frac{V}{r}$. In the branch P S Q, however, we have the opposing E.M.F. of the battery, so that the current which passes from P to Q along P S Q is $\frac{V - E}{b}$. Now, since $E > V$, it follows that $V - E$ will be negative—*i.e.*, the current will, as we know, pass from Q to P, and not from P to Q, although P is at a higher potential than Q.

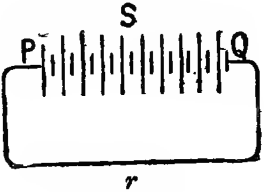


FIG. 15.

To take another example, let us suppose that a dynamo is charging a set of secondary cells. Let V be the potential difference at the terminals of the cells, E their E.M.F., and b their resistance. The current going into the cells is $i = \frac{V - E}{b}$. So long as $V > E$, $V - E$ is positive, and the cells are being charged; but if (owing, say, to a lowering of the speed of the dynamo) V becomes less than E, $V - E$ will be negative, and the cells will discharge through the dynamo, driving it as a motor.

The above examples will suffice to show that in applying Ohm's law to any circuit, or part of a circuit, a certain amount of discretion must be used. The whole of the E.M.F.'s present in the circuit, or that part of it which is being considered, must be taken into account.

26. As a final example, we shall consider the case of a circuit *in which the current is varying.* As has been already explained in Art. 24, the variations of the current call into play E.M.F.'s of induction, and these have to be taken into account in determining the current. It is useful to draw a distinction between the E.M.F. which is the primary cause of the current and the induced E.M.F.'s which merely make their appearance when the current is undergoing variations. The first is generally termed the *impressed* E.M.F. At any instant, the following relation holds:

$$\text{current} = \frac{\text{impressed E.M.F} + \text{induced E.M.F.}}{\text{resistance.}} \qquad (15)$$

It is the *algebraical* sum of the E.M.F.'s that is understood in the numerator of the fraction on the right-hand side of (15). The induced E.M.F. may be either of the same sign as the impressed E.M.F., or of opposite sign. The relation expressed by (15) is an extremely important one, and we shall frequently have occasion to use it.

27. It is necessary to say a few words with regard to the *power* in a circuit containing several E.M.F.'s. To begin with, suppose we have a circuit of resistance r, in which there is only one E.M.F., say E. The current is $i = \frac{E}{r}$. The rate of expenditure of energy is $E\,i$, or $i^2 r$. The latter represents the rate at which heat is being produced in the circuit, and in this case the whole of the energy which is supplied by the source of E.M.F. is converted into *heat*. Next, suppose the circuit to contain an opposing E.M.F., or counter E.M.F., of amount e. The current will now be $i_1 = \frac{E - e}{r}$, whence $E = i_1 r + e$. The power is $E\,i_1 = i_1^2 r + e\,i_1$. The first term, $i_1^2 r$, represents as before, the rate at which dissipation of energy takes place by heating of the circuit; the second term, $e\,i_1$, represents the rate at which energy is being absorbed by the source of counter E.M.F. This energy may either be *stored up* (as in the case of secondary cells), or it may be converted into any other form of energy which undergoes immediate further transformations (as in the case of an electromotor). The whole of the power in the circuit may therefore be regarded as made up of two parts: one part going towards producing *heat* in the circuit, the other towards the transforma-

tion of the electrical energy into forms *other than that of heat.* When the current in a circuit is increasing, the counter E.M.F. induced in the circuit causes *storage of energy*, the form which the energy takes being that of a magnetic field. The magnetic field due to a current, in fact, represents so much potential energy, and we shall see presently how this energy may be determined.

Considering next the most general case, in which we have a number of E.M.F.'s in a circuit of resistance r, let us suppose that the E.M.F.'s whose direction is the same as that of the current, i, are denoted by $E_1, E_2, E_3 \ldots$, and the E.M.F.'s which are opposed to the current by e_1, e_2, e_3, etc. The E.M.F.'s E_1, E_2, E_3, etc., act as *sources* of energy; the E.M.F.'s e_1, e_2, e_3, act as absorbers or *sinks* of energy. The *current* in the circuit may be regarded as transferring energy from the sources to the sinks. But this transference cannot be accomplished without a certain amount of the energy being dissipated in the form of heat, the rate of dissipation being given by $i^2 r$. We thus have the equation:

$$\underbrace{(E_1+E_2+E_3+\ \ldots)\ i}_{\text{Total rate of production of energy}} = \underbrace{(e_1+e_2+e_3+\ldots)\,i}_{\text{rate of absorption}} + \underbrace{i^2 r}_{\text{rate of dissipation.}}$$

28. Consider a simple circuit conveying a current. A certain number of magnetic lines will thread themselves through the circuit, as shown in Fig. 16. It must be understood that the lines which we are considering are merely *those due to the current* in the circuit, and do not include any lines which may be due to the presence of magnets or other circuits

conveying currents. We also suppose that the medium surrounding the circuit is air. Besides the magnetic lines which pass through the space surrounding the conductor, we have a number of lines in the substance of the conductor itself. In Fig. 17 let the thick-line circle represent the section of the wire forming the circuit. The thin-line circles are supposed to represent the magnetic lines *due to the current.** At the centre of the circle* the magnetic force is zero. As we proceed from the centre outwards the magnetic force gradually *increases,* reaching a maximum just at the boundary of the conductor; on receding from the boundary of the wire we find that the magnetic force *decreases.* In the case of many circuits, the magnetic lines which are confined to the substance of the conductor are very few in comparison with those traversing the space outside. Cases may arise, however (such as that of a circuit formed by two

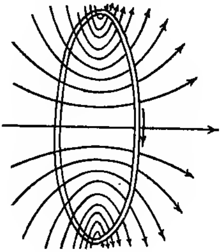


FIG. 16.

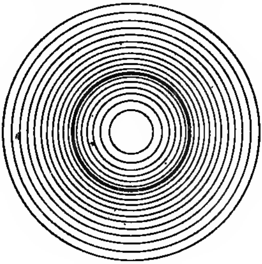


FIG. 17.

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* This is only an approximation. It would be strictly correct for an infinitely long wire at a distance from all other conductors conveying currents.

parallel iron wires placed a short distance apart), in which the magnetic lines inside the conductor are quite as important as those outside.

We may imagine the conductor forming the circuit shown in Fig. 16 to be split up into a large number of parallel filaments of very small cross-section. It is evident that the number of magnetic lines enclosed by a filament will depend on its position. A reference to Fig. 17, which gives the approximate distribution of the lines in the conductor and the space immediately surrounding it, shows that the axial filament encloses more lines than any other. Hence, if we suppose that the current flowing round the circuit is made to vary, the E.M.F. induced along each filament will depend on its position, the greatest value of the induced E.M.F. occurring along the axis of the wire, and its least value along the filaments forming the surface of the wire. The average value of the induced E.M.F. over the cross-section of the conductor is intermediate between its values for the axial and one of the surface filaments respectively. The average E.M.F. is thus measured by the rate of change of the quantity :

$$N_0 + n,$$

where N_0 stands for the total number of lines traversing the space *outside* the conductor, and n for a certain number of lines which is *less* than the number *inside* the conductor. We shall call N_0+n the *magnetic flux* through the circuit. It must be clearly understood that the magnetic flux is intermediate in value between the total number of lines produced by the current and the number of lines in the space outside the conductor forming the circuit.

Some important results may at once be deduced from the above considerations. If we suppose that the current is made to undergo variations, then, since (Art. 24) the E.M.F.'s induced by such changes of current always oppose the changes, and since, as we have just seen, these opposing E.M.F.'s are *less* for the outer portions of the conductor than for the inner ones, it follows that the former will respond *more quickly* to changes of current than the latter. In other words, during changes of current in the circuit there will be an unequal distribution of current over the cross-section of the conductor. Such effects will, however, be fully considered later on (Arts. 108-111), and we merely draw the student's attention to them at this stage.

29. Since we have supposed the circuit to be surrounded by air, it follows* that the magnetic flux through it will be directly proportional to the current. Hence, if F stand for the magnetic flux due to a current i, we may write

$$F = Li \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

where L is a constant which is independent of i. The constant L is termed the *coefficient of self-induction,* or *inductance,* of the circuit. Since F becomes equal to L when $i=1$, we may define the *inductance of a circuit* as *the magnetic flux through it due to unit current.*

---

* The conductor itself being also supposed to consist of some non-magnetic material.

We have next to explain the meaning of the term "inductance" when applied to a complex circuit such as a coil of wire. The circuit represented in Fig. 18 is supposed to be made up of three loops. In order to determine the magnetic flux due to unit current, we may imagine the complete circuit split up, as in Fig. 19, into a number of simpler circuits. The sum of the magnetic fluxes through the various simple circuits into which the complex circuit has been broken up gives us the total magnetic flux due to unit current, and this we define to be the *inductance* of the complex circuit.

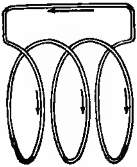

FIG. 18.

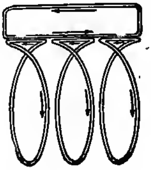

FIG. 19.

30. If we imagine a coil the diameter of which is large compared with the dimensions of its cross-section, we can easily deduce an approximate expression for the inductance of the whole coil in terms of the inductance of a single turn. For in the case of such a coil we may roughly assume that the magnetic flux through each turn will be the same. Let l stand for the inductance of a single turn of the coil, and let there be S turns in the coil. Then, since a unit current through one turn causes l lines to pass through that turn, a unit current through the

whole coil—*i.e.*, through S turns—will cause S l lines to pass through *each* turn. Hence the total magnetic flux through the coil when conveying unit current—*i.e.*, its *inductance*—will be $S \times S\, l = S^2 l$. We thus arrive at the result that the inductance of a coil varies approximately as the *square of the number of turns* in it.

31. In the case of some circuits, such as a wire which is doubled back upon itself, the inductance is extremely small. Such circuits are usually called *non-inductive* circuits. It is impossible to obtain a circuit which is absolutely non-inductive, but the term is applied to circuits whose inductance may, for all practical purposes, be left out of account. By an *inductive* circuit we mean one whose inductance is appreciable. It is convenient to have special symbols for denoting non-inductive and inductive resistances, the symbols in general use being those shown in Fig. 20.

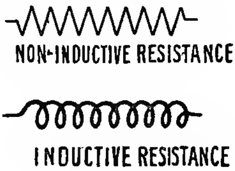

Fig. 20.

A common form of non-inductive resistance consists of a wire doubled on itself. The magnetic field due to such an arrangement is represented in Fig. 21. The points of zero magnetic force O and O_1 are not at the centres of the two conductors (the outlines of whose sections are indicated by the thick-line circles), but are displaced outwards. Such an arrangement produces an external field, which, however, becomes extremely feeble as the distance from the conductors increases.

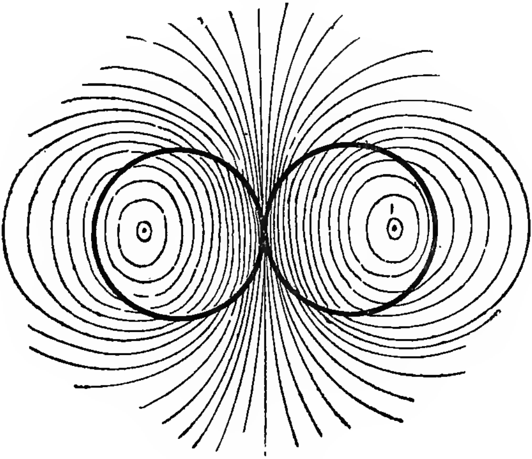

Fig. 21.

A more perfect arrangement is obtained by making one of the conductors a hollow cylinder which completely encloses the other. The lines of magnetic induction in this case consist of concentric circles, as shown in Fig. 22, and there is absolutely no field outside the conductors, the lines being entirely confined to the substance of the conductors (and the thin layer of insulation between them). Figs. 21 and 22 are both drawn roughly to scale, the conductors being supposed to convey the *same* current in each case. By counting the number of lines in the two cases, the student will see that the arrangement shown in Fig. 21 gives rise to about three times as many lines as that shown in Fig. 22.

For all practical purposes, two conductors (of non-magnetic material) laid side by side—the one conveying the outgoing current and the other the return—may be regarded as forming a non-inductive arrangement.

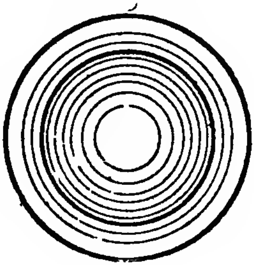

FIG. 22.

In many cases, in fact, a conductor forming a *narrow* loop (such as the filament of an incandescent lamp) may be considered as a non-inductive circuit.

32. Let us suppose that a steady E.M.F. is suddenly applied to an inductive circuit of resistance r. The current will not *suddenly* rise to its steady value, $\frac{E}{r}$, but will take an appreciable (though in most cases very short) time to do so. This is due to the fact, already explained, that as soon as the current begins to rise a varying magnetic flux is introduced into the circuit, and

this varying flux induces an E.M.F. which opposes the current. While the current is growing, energy is being absorbed by the circuit in other ways than that of heat (Art. 27). What becomes of this energy? The answer is, that it becomes stored up in the form of a magnetic field surrounding the circuit. We shall now show how this energy may be calculated if the inductance L of the circuit is known.

Suppose N magnetic lines to be introduced during t seconds into a circuit in which a steady current of i C.G.S. units is maintained. The counter E.M.F. induced by the insertion of the lines is (Art. 23) $\frac{N}{t}$; hence the rate at which work is being done against the induced E.M.F. is (Art. 27) $\frac{N}{t} . i$. The total amount of work done will therefore be $\frac{N}{t} \times i \times t = N i$ ergs. We thus have the following important result:

In increasing the magnetic flux through a circuit conveying a current i by N lines N i ergs of work are done.

From this we see that in order to build up the magnetic field of any circuit conveying a current, a definite amount of energy must be supplied to the circuit. We can easily determine this amount by the following method.

Taking the most general case of a circuit, in which the magnetic flux is *not* proportional to the current—*i.e.*, a circuit containing cores of magnetic material—let us suppose that the curve O B F D G (Fig. 23) gives the relation between the current in the circuit and the corresponding magnetic flux through the circuit. Let us suppose that at a certain instant the current has the value A B, and the corresponding flux the value O A. Let the current be made to increase by a *very small* amount, so as to assume the value C D; the magnetic flux will increase by the small amount A C. Since the length B D of the curve is supposed to be very small, we may regard it as a short straight line. The *mean* value of the current between A B and C D will then be given by E F, where E is half-way between A and C. Instead of supposing that the current steadily increases from A B to C D, we may assume that it has the constant mean value E F while the flux increases by A C. The work done in increasing the flux will then be, by the proposition just proved, A C × E F = area of A B D C, so that in increasing the current from A B to C D, and the flux from

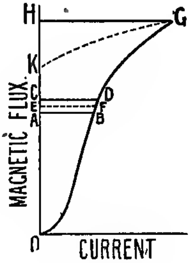

Fig. 23.

O A to O C, an amount of work is done which is represented by the area A B D C. The same result holds for every small increase of the current. Hence, the energy communicated to the magnetic field when the current rises from its zero value to the value H G is represented by the area O B G H.

In the case of a circuit whose permeability is constant, the flux is simply proportional to the current, and the relation between these two quantities will be represented by a straight line, as shown in Fig. 24. Let us suppose that the inductance of the circuit is L. In starting a current i = A B in the circuit, we store up an amount of energy in the magnetic field which is equal to the area of the triangle O B A—*i.e.*, to $\frac{1}{2}$ L i^2 (since A B = i, and O A = Li). Hence,

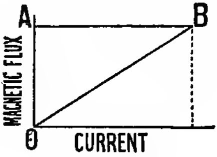

FIG. 24.

The energy stored up in the magnetic field of a circuit of inductance L conveying a current i is $\frac{1}{2}$ L i^2 ergs, L and i being measured in C.G.S. units.

When the current is arrested by breaking the circuit, the potential energy of the magnetic field makes its appearance in the form of a spark. In the case of a circuit whose permeability is constant, the *whole* of the energy is returned at the instant of breaking the circuit, since all the magnetic lines which were produced by the current disappear on breaking the circuit. But when the magnetic flux passes through a substance exhibiting *hysteresis*, part only of the energy is recoverable. Thus, if in Fig. 23 the dotted curve G K represent the descending branch of the hysteresis curve, the energy which is returned by the circuit when the current is arrested is represented by the area G K H. The area O B G K represents the energy dissipated *in the magnetic substance* by heating of the latter. If we suppose the magnetic substance carried through a complete cycle, we see that *the energy dissipated is represented by the area of the hysteresis loop.* The abscissæ of the loop, it must be noted, represent *current*, and the ordinates magnetic flux.

Since the energy stored up in a circuit of given inductance is proportional to the *square* of the current, it will be readily seen why the destructive effect of a spark increases very rapidly with the current. Thus, by increasing the current tenfold, the destructive effect of the spark is increased a hundredfold.

The *practical* unit of inductance is termed the *henry*, and it is equal to 10^9 C.G.S. units. In order, therefore, to determine the magnetic flux in C.G.S. lines through a circuit of inductance L *henries* conveying a current of i *amperes*, we must multiply the product $L\,i$ by 10^8 $(= 10^9 \times 10^{-1})$, and in order to determine the energy stored up in *ergs*, the product $\frac{1}{2}\,L\,i^2$ must be multiplied by 10^7.

In most cases occurring in practice the inductance of a circuit is expressed by a small fraction of a henry.

Notes on Chapter III.

On the subject of this chapter the student may consult Chapters I. and II. of "The Alternate-Current Transformer," Vol. I., by Dr. Fleming (*The Electrician* Printing and Publishing Company). He will also find it useful to refer to Chapter V. of Dr. Lodge's "Modern Views of Electricity" (Macmillan and Co.).

Exercises.

1. Two million C.G.S. lines of magnetic induction are withdrawn from a certain circuit at a constant rate during one-tenth of a second. Find the E.M.F., in volts, induced in the circuit.

2. A shunt-wound dynamo is charging a set of 50 secondary cells. The resistance of the leads and cells is ·5 ohm, and the counter E.M.F. of each cell is 2·3 volts at a certain instant. What must be the P.D. at the dynamo terminals in order to produce a current of 10 amperes?

3. A certain coil of wire has an inductance of ·02 henry, and the current passing through it is at a certain instant found to be changing at the rate of 100 amperes per second. Find the E M.F., in volts, induced in the coil at that instant.

4. Two bobbins of the same size and shape are wound with different sizes of wire, so that one of them contains thrice as many turns as the other. The inductance of the bobbin with the smaller number of turns is found to be ·008 henry. Find the energy, in foot-pounds, stored up in the magnetic field of the other bobbin when conveying a current of 2 amperes.

CHAPTER IV.

33. We are now in a position to consider the *growth* of a current in an inductive circuit under a steady impressed E.M.F. At any instant, the impressed E.M.F. may be regarded as made up of two components—viz., the component required to maintain the current through the resistance of the circuit, and the component required to balance the counter E.M.F. due to the inductance of the circuit. The first component is simply the product of the current into the resistance; we shall refer to it as the $r\,i$ component of the E.M.F. The second component is measured by the rate of change of the magnetic flux through the circuit. Let us suppose that the inductance of the circuit is L, and that during t seconds the current has changed from i_1 to i_2; the corresponding change in the magnetic flux is $\mathrm{L}\,i_2 - \mathrm{L}\,i_1 = \mathrm{L}\,(i_2 - i_1)$. The mean *rate of change* of the magnetic flux during the same time is $\frac{\mathrm{L}\,(i_2 - i_1)}{t} = \mathrm{L}\,\frac{i_2 - i_1}{t}$; but $\frac{i_2 - i_1}{t}$ gives us the rate of change of the current. Hence rate of change of magnetic flux $=$ L × rate of change of current.

It is convenient to use a special symbol to denote the rate of change of the current at any instant, and we shall use i' to denote this rate of change.

If we suppose E to denote the E.M.F. applied to the circuit, we have the relation

$$E = ri + Li' \quad . \quad . \quad . \quad . \quad (17)$$

This equation will hold good whether E be constant or not. In the present case we suppose E to be constant, but cases will present themselves later on in which E is variable. The equation expresses the fundamental relation between the instantaneous values of the E.M.F. and current in an inductive circuit, and it is very important that the student should have a perfectly clear conception of the physical meaning of the equation. E, i and i' may all change from instant to instant; but *at any one instant* these three quantities are so related as to satisfy equation (17).

34. From equation (17) we have at once

$$i' = \frac{E - ri}{L} \quad . \quad . \quad . \quad . \quad (18)$$

This equation enables us to calculate i' for any given value of i. The relation between these quantities may be represented by a straight line, as shown in Fig. 25. The greatest rate of change of the current, O A $\left(= \frac{E}{L}\right)$, occurs just at the commencement, when the value of the current is zero. As the current increases, the *rate* at which it does so diminishes, until finally, when the current has reached its steady value, O B $\left(= \frac{E}{R}\right)$, the rate of change becomes nothing.

The student will readily see that the rapidity with which the current rises to any fraction of its steady value will depend on the inclination of the line B A in Fig. 25 to the vertical. The greater this inclination the more rapidly will the current establish itself in the circuit. For if we suppose that the line B A is displaced into the position B A′ so as to become more inclined to B O, then for any given value of the current such as O P, its *rate of growth* is *increased* from P Q to P Q′; hence, the current will attain its steady value more rapidly in the second case. *Any change in the circuit, therefore, which causes the point A to travel away from the origin will accelerate the rise of current in the circuit.* Now, for a given value of the steady current, $\frac{E}{r}$, the distance of A from the origin may be increased in two ways: first, by *diminishing* L; secondly, by increasing *both* E and r, so that the ratio $\frac{E}{r}$ remains unaltered. In most cases which present themselves in practice it is not practicable to reduce L, but we may in most

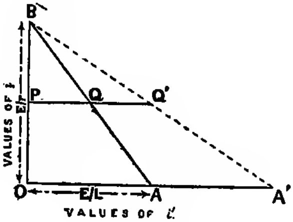

FIG. 25.

cases adopt the second expedient of connecting in series with the inductive resistance a large non-inductive one (which adds practically nothing to L but materially increases r), and using a higher E.M.F.

35. Since the rapidity with which a current establishes itself in a circuit depends on the inclination of the line B A, it is convenient to know what the angle O A B is. This angle is generally assigned in terms of its tangent (Art. 2), and the tangent is termed the *time-constant* of the circuit. Now $\tan \mathrm{OAB} = \frac{\mathrm{OB}}{\mathrm{OA}}$—*i.e.*, since $\mathrm{OB} = \mathrm{E}/r$, and $\mathrm{OA} = \mathrm{E}/\mathrm{L}$, $\tan \mathrm{OAB} = \frac{\mathrm{L}}{r}$, so that we have the following alternative definition :

The time-constant of a circuit is the quotient of its inductance by its resistance.

In a circuit having a large time-constant, the current takes a comparatively long time to rise to its steady value; in a circuit whose time-constant is small, it establishes itself very rapidly.

The diagram of Fig. 25 may be used for an approximate method of determining the relation between the current and the *time*. The method is based on the assumption that the rate at which the current changes does not vary continuously, but by a series of small steps. The student will best understand the method by studying the following numerical example.

36. Let an E.M.F. of 100 volts be applied to a circuit whose resistance is 10 ohms and inductance

·05 henry. The value of the steady current is $\frac{100}{10}$ = 10 amperes. In Fig. 26, let O be taken as origin, and let the vertical axis be the axis of current. We shall suppose that the time is measured along the horizontal axis to the right of the origin. Lay off a length O B = 10 units to represent, on a convenient scale, the 10 amperes. From O lay off O A $= \frac{E}{L} = \frac{100}{\cdot 05} = 2{,}000$, to a convenient scale. The line O A in Fig. 26 corresponds to the line O A in Fig. 25, and it represents the rate at which the current increases just at the commencement. Divide O B into any number of equal parts, say 20; each of these divisions will therefore represent ·5 ampere. The greater we make the number of these divisions the more accurate will be the result. Through the middle point of each division draw a line parallel to O A until it intersects the line A B. The first of these lines determines the mean* rate of variation

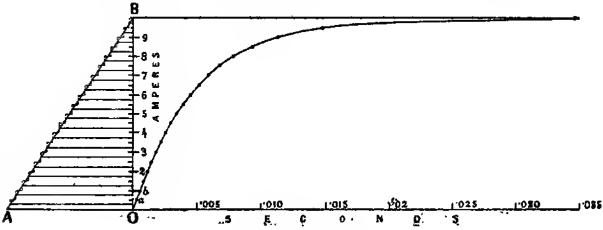

FIG. 26.

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* Mean, that is to say, with respect to the current, but *not* with respect to the *time*; and that is precisely where the error is introduced in this approximate method.

of the current between the values 0 and ·5 ampere, and we shall suppose that as the current increases from nothing up to ·5 ampere, its rate of change, instead of varying, remains constant and equal to the mean rate of change between those limits. When the current has reached the value ·5, we suppose that its rate of change *suddenly* diminishes, and remains constant while the current increases from ·5 to 1 ampere, and equal to the mean rate of change between those values of the current, and so on. This supposition is equivalent to assuming that the line A B, instead of being a continuous straight line, is a stepped or zig-zag line (shown dotted in Fig. 26).

In order to construct the curve connecting *time* and *current*, we proceed as follows. From the diagram to the left of O B we find that the mean rate of change of the current between 0 and ·5 ampere is 1,950 amperes per second. This means that the current would change by 1 ampere in $\frac{1}{1950}$th, or ·000513, of a second; hence it would increase by ·5 ampere in ·000256 second. We thus get one point on our curve (*a* in Fig. 26). To obtain the next point, we notice that between ·5 ampere and 1 ampere the mean rate of increase of the current is 1,850 amperes per second. Hence the current would increase from ·5 ampere to 1 ampere—*i.e.*, by half an ampere, in $\frac{1}{1,850 \times 2}$ = ·00027 second. Corresponding, therefore, to one ampere we have the time ·000256 + ·00027 = ·000526 second. This gives us another point, *b*, on our curve. By proceeding in this manner we obtain 20 points. According to the supposition which we have made, the growth of the current would be represented by the open polygon formed by joining the points determined as above by straight lines. But we know that the rate of change of the current varies *continuously*, and not by sudden steps. Hence it will be more correct to draw a continuous curve through the points. This curve is shown in Fig. 26.

The approximate method just explained for drawing the *time-current* curve presents no difficulties, and is capable of furnishing very accurate results. The accuracy may be increased as much as we please by sufficiently increasing the number of points on the curve.

As a matter of fact, the current never reaches the value which we call the "steady value" $\left(\frac{E}{r}\right)$. But after a very short time from the instant of closing the circuit it differs so very little from its steady value, and grows so very slowly, that for all practical purposes it may be regarded as having reached the steady value. Thus, in the example considered, it may be shown that the current falls short of its steady value by only $\frac{1}{100}$th of 1 per cent., ·046 second after closing the circuit.

37. We now pass on to the consideration of an inductive circuit conveying an alternating current which is a simple harmonic function of the time. We shall suppose that the resistance, r, and inductance, L, of the circuit are known, and that the maximum value of the current is I. The problem is to find the impressed E.M.F. (Art. 26) in the circuit.

At any instant, if i denote the current, we have the following relation (Art. 33):

$$\textit{impressed}\ \text{E.M.F.} = r\,i + \text{L}\,i',$$

where i' stands, as before, for the rate of change of the current at the instant considered. The impressed E.M.F. is thus seen to be made up of two components, the component $r\ i$, which serves to maintain the current through the *resistance* of the circuit, and the component L i', which balances the E.M.F. called into play by the *inductance* of the circuit. Since we suppose the current to be a simple harmonic function of the time, we may represent its value at any instant (Art. 22) by the projection on the vertical axis of a rotating vector of length I. By taking another rotating vector, O A (Art. 27), of length r I (the vector being in the same phase as that which represents the current) and projecting it on the vertical axis, we obtain an alternating vector whose value at any instant gives us the $r\,i$ component of the impressed E.M.F. Again, since the current is capable of being represented by an alternating vector, its rate of change will be represented by another alternating vector (Art. 22), whose amplitude is p I (p standing for the angular velocity), and which is $\frac{\pi}{2}$ (or 90deg.) in advance of it as regards phase. The rate of change of the current would therefore be obtained by projecting a rotating vector of length p I, and at right angles to O A, on the vertical axis; and the L i' component of the E.M.F. (= L × rate of change of current) may be represented by the projection on the vertical axis of a rotating vector O B (Fig. 27) of length L p I, which is at right angles to O A. We thus see that the impressed E.M.F. is the sum of two alternating vectors. In order to obtain this sum of the projections of O A and O B, we may first find the resultant O C of the two rotating vectors O A and O B (Art. 18 and Fig. 11). The projection of the rotating vector O C on the vertical axis gives us the instantaneous value of the impressed E.M.F.

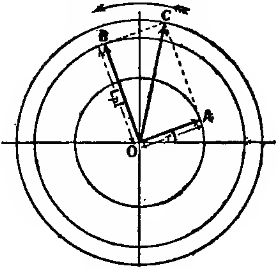

FIG. 27.

38. We have thus shown that if the current in a circuit of constant inductance obeys the simple harmonic law, the impressed E.M.F. is also a simple sine function of the time. In order to find the maximum value O C of this E.M.F., we have merely to notice that since the triangle A O C is right-angled,

$$\begin{aligned} \mathrm{O\,C} &= \sqrt{\mathrm{O\,A}^2 + \mathrm{A\,C}^2}, \\ &= \sqrt{\mathrm{O\,A}^2 + \mathrm{O\,B}^2}, \\ &= \sqrt{r^2\,\mathrm{I}^2 + \mathrm{L}^2\,p^2\,\mathrm{I}^2}, \\ &= \mathrm{I}\,\sqrt{r^2 + \mathrm{L}^2\,p^2}. \end{aligned}$$

From Fig. 27 we also see that the current is not in phase with the impressed E.M.F., but *lags* behind it by an angle A O C. Now

$$\tan \mathrm{A\,O\,C} = \frac{\mathrm{A\,C}}{\mathrm{O\,A}} = \frac{\mathrm{L}\,p\,\mathrm{I}}{r\,\mathrm{I}} = \frac{\mathrm{L}\,p}{r},$$

i.e., the tangent of the angle of lag is equal to the quotient of $\mathrm{L}\,p$ by r.

If we now suppose that a simple harmonic E.M.F., $\mathrm{E} \sin p\,t$, is applied to a circuit of resistance r, and inductance L, then it follows from the above that the current is given by

$$i = \frac{\mathrm{E}}{\sqrt{r^2 + p^2\,\mathrm{L}^2}} \,.\, \sin\,(p\,t - \theta). \quad . \quad (19)$$

where θ is an angle such that its tangent is equal to $\frac{p\,\mathrm{L}}{r}$.

If we use I to denote the maximum value of the current, then

$$\mathrm{I} = \frac{\mathrm{E}}{\sqrt{r^2 + p^2\,\mathrm{L}^2}} \quad . \quad . \quad . \quad . \quad (20)$$

These equations are of fundamental importance in the theory of alternating currents, and should be committed to memory by the student.

Equation (20) shows that the maximum value of the current is obtained by dividing the maximum value of the E.M.F. by the quantity $\sqrt{r^2 + p^2\,\mathrm{L}^2}$. This quantity, which is the analogue of resistance in a continuous-current circuit, is termed the *impedance* of the circuit. The quantity p L is called the *reactance*. We thus have

$$(impedance)^2 = (resistance)^2 + (reactance)^2 \, ;$$

and if θ denote the angle of lag of the current behind the E.M.F., we have

$$\tan\,\theta = \frac{reactance}{resistance} \quad . \quad . \quad . \quad . \qquad (21)$$

The diagram (Fig. 28), which a reference to Fig. 27 will at once enable the student to understand, may help to exhibit the important relations between the quantities we have just been considering.

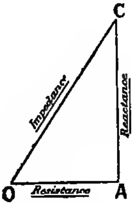

Fig. 28.

39. The variations of an alternating E.M.F. or current which is a simple sine function of the time (Art. 22) are always capable of being represented by a simple sine curve, such as that which is shown in Fig. 12. The time taken by the E.M.F. or current to run through a complete cycle of values is, as has already been explained (Art. 14), termed a *period*. The number of complete periods in one second is termed the *periodicity*. The symbol ∿, introduced

by Mr. Mordey, is used to denote the periodicity. Thus 100 ⌒⌣ per second means "100 complete periods in every second." If the periodicity n is given, then, remembering that $p = 2\,\pi\,n$ (Art. 17), we can at once calculate the reactance, and hence also the impedance, for a circuit whose resistance and inductance are known. The periodicities commonly used in this country vary from 40 to 100 ⌒⌣ per second.

40. We shall now apply the results established above to a numerical example.

Let an alternating simple harmonic E.M.F. whose maximum value is 100 volts be applied to an inductive circuit whose resistance is 18·2 ohms, and inductance ·0796 henry. The periodicity we shall suppose to be 100 ⌒⌣ per second. In order to find the maximum value of the current we must calculate the impedance. Now $p = 2\,\pi \times 100 = 628{\cdot}32$, so that

$$\text{reactance} = p\,\text{L} = 628{\cdot}32 \times {\cdot}0796 = 50.$$

Hence

$$\text{impedance} = \sqrt{r^2 + p^2\text{L}^2} = \sqrt{18{\cdot}2^2 + 50^2} = 53{\cdot}21.$$

Therefore,

$$\text{maximum value of current} = \frac{\text{maximum value of E.M.F.}}{\text{impedance}}$$

$$= \frac{100}{53{\cdot}21} = 1{\cdot}88 \text{ amperes.}$$

Next, if θ denote the angle of lag,

$$\tan\theta = \frac{\text{reactance}}{\text{resistance}} = \frac{50}{18{\cdot}2} = 2{\cdot}747.$$

In order to determine θ, we may, as in Fig. 28, construct a right-angled triangle O A C whose sides O A and A C are equal to the resistance and reactance respectively. The angle A O C will then be the angle of lag. By measuring this angle with a protractor the student will find it to be about 70deg.

To construct the E.M.F. and current curves we may adopt the method explained in Art. 20 and illustrated by Fig. 12. Take two rotating vectors, O A (Fig. 29), of length = 100 (to a convenient scale), and O B, of length = 1·88, inclined to each other at an angle

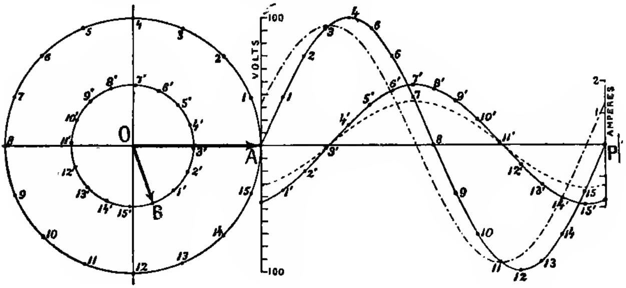

FIG. 29.

of 70deg. As the two vectors rotate with the same angular velocity, p (= 628·32 radians per second), round O, the projection of O A on the vertical axis gives the instantaneous value of the E.M.F., while that of O B gives the instantaneous value of the current. By dividing the circumferences of the two circles along which the extremities of the rotating vectors travel into the same number of equal parts, and proceeding as explained in Art. 20, we derive the E.M.F. and current curves shown to the right of Fig. 29. The points of division 1, 2, 3 . . ., and 1′, 2′, 3′ . . ., are so related that they correspond to simultaneous positions of the two vectors. The distance A P corresponds to a period—*i.e.*, to ·01 second.

41. The impressed E.M.F., as we have seen, may be regarded as made up of two components, the one to overcome the resistance of the circuit (the $r\,i$ component), the other to balance the effect due to the induced E.M.F. We can readily analyse the impressed E.M.F. curve shown in Fig. 29 into these two components. By taking a number of points on the current curve, and multiplying the corresponding ordinates by 18·2 (the resistance of the circuit), we obtain the $r\,i$ component of the E.M.F., shown dotted in Fig. 29. By subtracting the ordinates of the curve so obtained from the impressed E.M.F. curve, we obtain another sine curve, which gives us the component required to balance the E.M.F. due to inductance. This is shown chain-dotted in the figure. These curves might also have been obtained directly by projecting two rotating vectors, the one of length $r\,I$ and in phase with O B, the other of length $L\,p\,I$ and at right angles to O B (*c f*, Fig. 27). The student will notice that the $L\,i'$ component of the impressed E.M.F. passes through its zero value when the current reaches its maximum value—*i.e.*, becomes stationary for the moment, so that its rate of change is nothing—and *vice versâ*. This will also be at once evident by referring to Fig. 27.

42. The quantity, then, which determines the amplitude of the current—*i.e.*, its maximum value, Art. 14—in an inductive circuit is not merely the

resistance of the circuit, but its *impedance*. If the periodicity is low, then the resistance of the circuit is more important than its reactance, so that for *very slow* alternations of the E.M.F. the current will depend mainly on the *resistance* of the circuit. As the periodicity increases, however, the *reactance* becomes more and more important, and for *extremely rapid* alternations of the E.M.F. the impedance will depend almost entirely on the *reactance*, the resistance becoming practically insignificant.

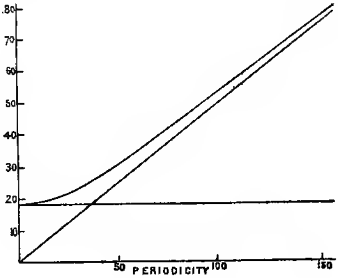

FIG. 30.

In order to exhibit this change of impedance with the periodicity the numerical values of the reactance and impedance corresponding to various periodicities for the circuit already considered have been plotted in Fig. 30. The horizontal straight line above the axis of periodicity, and at a distance of 18·2 from it, represents the *resistance* of the circuit, which is assumed to be independent of the periodicity.* The sloping straight line represents the *reactance*, which increases at a constant rate with the periodicity. Finally, the uppermost curved line gives the values of the *impedance*, calculated from the formula $\sqrt{r^2+p^2L^2}$. It will be noticed that for *very low* values of the periodicity, the impedance line is almost coincident with the line of resistance, while for *very high* values it tends more and more towards coincidence with the line of reactance.

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* As a matter of fact, the resistance is *not* independent of the periodicity, owing to the crowding of the current towards the outer portions of the conductor. This effect will be discussed later.

We have hitherto supposed that we are dealing with a circuit for which L is a constant. In many important practical cases, L is a variable quantity; this is always the case for any coil containing a core of magnetic material. The simple relations which we have established between the E.M.F. and current will then no longer hold. But the general principles involved will be the same. Thus, the fundamental equation connecting impressed E.M.F. and current will assume the form

impressed E.M.F. $= ri +$ *rate of change of magnetic flux*,

instead of the more special and simpler form of Art. 33. The general conclusions regarding the effects of resistance, reactance, and impedance which we have deduced for a circuit of constant permeability will also hold for a circuit containing iron.

Notes on Chapter IV.

Equation (20) was first given by Helmholtz, and is sometimes referred to as Helmholtz's equation. The time taken by a current to reach a value which for all practical purposes is its "steady" value, is generally a small fraction of a second; in some cases, however, where the inductance is exceptionally large, several seconds may lapse before the current becomes practically steady. The rate at which a current grows in a circuit under a constant E.M.F. is a matter of very great importance in fast-speed telegraph instruments, and special devices are used in such cases to accelerate the growth of the current.

Exercises.

1. The resistance of a circuit is ·5 ohm, and its inductance is ·001 henry. Draw a curve showing the rise of current in the circuit.

2. The resistance of an armature coil is 5×10^{-4} ohm, and its inductance 10^{-5} henry. The coil is conveying a current of 200 amperes. As it passes under the brush it is short-circuited, and a reversing E.M.F. of 2 volts is applied to it. How long must the coil remain short-circuited before a reversal of current takes place? Ans.: ·002 second.

3. The resistance of a circuit is 2 ohms, and its inductance ·02 henry. Find how long the current will take to reach 80 per cent. of its steady value.

4. If an additional non-inductive resistance of 8 ohms is introduced into the circuit mentioned in the last example, how long will the current take to reach 80 per cent. of its steady value?

5. Find the maximum value of the current and the angle of lag in a circuit of resistance = 4 ohms, inductance = ·04 henry, when a simple harmonic E.M.F. of maximum value 100 volts and periodicity 80 is applied to the circuit.

6. A non-inductive circuit has a resistance of 2 ohms. How much inductance must be introduced into the circuit in order that the maximum value of the current in it may not exceed 5 amperes when a simple harmonic E.M.F. of amplitude = 100 volts, and periodicity = 100 $\sim$ per second, is applied to the circuit?

CHAPTER V.

43. We have hitherto supposed that the alternating E.M.F., or current with which we have to deal, is a simple sine function of the time. In many practical cases, the shape of the E.M.F. or current curve deviates more or less from the simple sine form. In some cases, such as that represented in Fig. 31, the deviation from the simple sine form is very slight; in others, again, such as that shown in Fig. 3, or the one in Fig. 33, the deviation is considerable. A knowledge of the shape of the E.M.F. wave of any given alternator is of considerable importance. The best shape of E.M.F. wave depends on the kind of work which the alternator has to do. Thus, a wave which would be very suitable for supplying trans-

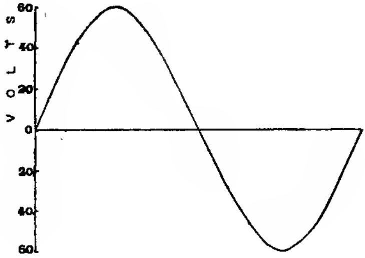

FIG. 31.

formers might prove totally unsuitable for supplying arc lamps. There is therefore no such thing as *the* best shape of E.M.F. wave for all purposes; whether a given shape is good or bad, depends on the nature of the apparatus to be supplied.

The results which we have already established for simple sine waves may be extended to waves of any shape. For this purpose, however, we must be able to *analyse* any given wave into its *simple harmonic components*.

It may be shown that a periodic function is capable of being split up or analysed into a number of simple harmonic terms, whose periodicities are whole multiples of the periodicity of the given function. Thus, if n stand for the periodicity—*i.e.*, the number of periods per second—of the function, the periodicities of its simple harmonic components will be n, $2n$, $3n$, $4n$, and so on. If F denote the function, then

$$F = F_1 \sin(pt + \theta_1) + F_2 \sin(2pt + \theta_2) + F_3 \sin(3pt + \theta_3) + \dots$$

where p stands for $2\pi n$.

This expansion of a periodic function in a series of simple harmonic terms is due to Fourier, and is known as Fourier's expansion.

The number of terms in the expansion will depend on the amount by which the function deviates from the simple sine form. In many cases an infinite number of such simple harmonic terms would have to be taken to represent the function with *perfect accuracy*. But in practice the number of terms will depend on the degree of accuracy which it is desired to obtain. And just as in calculating the square root of 2, say, we only carry the calculation to as many decimal places as will ensure the required degree of accuracy, so in using Fourier's expansion we take only the first few terms and neglect all the remaining terms. In order to ascertain the degree of accuracy corresponding to a given number of terms, we have merely to reconstruct the function from these terms, and compare the resultant curve with the graph of the given function.

The expression given above for the harmonic expansion of a periodic function F may be thrown into a slightly different form. Since, by (8) of Art. 8,

$$F_1 \sin(pt+\theta_1) = F_1 \cos\theta_1 \sin pt + F_1 \sin\theta_1 \cos pt,$$

$$F_2 \sin(2pt+\theta_2) = F_2 \cos\theta_2 \sin 2pt + F_2 \sin\theta_2 \cos 2pt, \text{etc.},$$

we may, putting $F_1 \cos\theta_1 = A_1, F_1 \sin\theta_1 = B_1$; $F_2 \cos\theta_2 = A_2$, $F_2 \sin\theta_2 = B_2$, write the expansion in the form

$$F = A_1 \sin pt + B_1 \cos pt + A_2 \sin 2pt + B_2 \cos 2pt + \quad . \quad (25)$$

44. The E.M.F. curve of an alternator is a *symmetrical curve—i.e.*, the negative half-waves are an exact copy, with sign reversed, of the positive half-waves. This at once follows from the fact that the poles of an alternator are all alike. A want of symmetry in the E.M.F. wave could only be produced by giving alternate poles a different shape.

In Fig. 32 is represented a periodic function, along with the simple harmonic terms into which it may be analysed (shown dotted). The sum of the ordinates of the three dotted curves gives us the full-line curve, which is the resultant function. In the present case, the function is represented by the sum of the three terms,

$$10 \sin (pt + \theta_1) + 6 \sin (3p\,t + \theta_2) + \sin 5pt,$$

where $\tan \theta_1 = \frac{3}{4}$ and $\theta_2 = -\frac{\pi}{2}$.

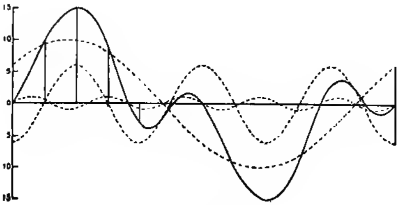

FIG. 32.

In Fig. 33 is represented the E.M.F. curve of a certain alternator. Let A and B be any two points the distance between which corresponds to a complete period. Divide A B into two half-periods, A C and C B. If now the portion of the curve included in the second half-period C B be superposed on that included in the first half-period A C, as shown by the dotted curve, it follows from the symmetry of the

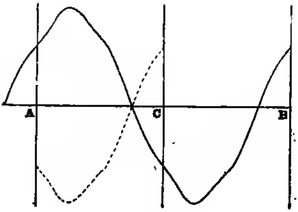

FIG. 33.

positive and negative half-waves that the resultant curve has every ordinate equal to zero. The two half-waves, therefore, into which any complete E.M.F. wave is divided will, when superposed, cancel each other. From this there follows at once the important result that *an alternator E.M.F. curve contains harmonic components of an odd order only, and (so long as it is symmetrical) can contain no harmonic component of an even order.*

To prove this, let us suppose that a certain periodic function contains a harmonic component of an even order. Let this harmonic component be represented by the full-line curve shown in Fig. 34, and let K L correspond to a complete period of the given function. If we divide K L into two half-periods K M and M L and superpose them, then, since each half-period contains a whole number of periods (or an *even* number of half-periods) of the harmonic component under consideration, the resultant curve (shown dotted in Fig. 34) will be one whose ordinates are double those of the given harmonic component. Hence no *symmetrical* curve can contain a harmonic component of an even order, for, as we have seen above, the

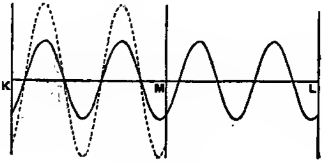

FIG. 34.

superposition of two half-periods of such a curve gives a resultant all of whose ordinates are equal to nothing.

If, on the other hand, a periodic function contains a harmonic component of an *odd* order, as shown in Fig. 35, then the superposition of two half-periods causes that component to vanish, for each half-period of the given function now contains an *odd* number of half-periods of the harmonic component considered.

Having thus shown that an alternator E.M.F. curve contains harmonic components (or *harmonics*) of an odd order only, we need not further consider curves

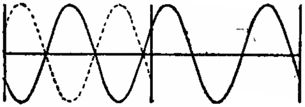

FIG. 35.

involving even harmonics. A function containing only odd harmonics may be written in the form

$$\begin{aligned} F = A_1 \sin p\,t + B_1 \cos p\,t + A_3 \sin 3\,p\,t \\ + B_3 \cos 3\,p\,t + A_5 \sin 5\,p\,t + B_5 \cos 5\,p\,t + \text{etc.} \end{aligned}$$

45. We shall next consider the distortion of shape produced in a simple harmonic wave by the superposition on it of a third harmonic. The shape of the resultant curve will depend very largely on the phase relation of the two curves. If the two curves are coincident in phase when the fundamental harmonic passes through its zero value, as in Fig. 36, the resultant curve is flattened at the top. If, on the other hand, the two curves are in opposition as

regards phase, as in Fig. 37, the resultant curve runs up to a sharp peak. In both cases each half-wave

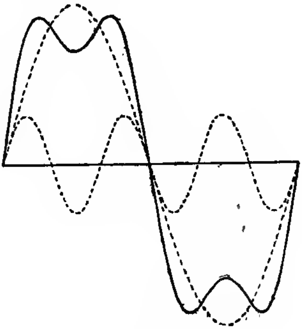

Fig. 36.

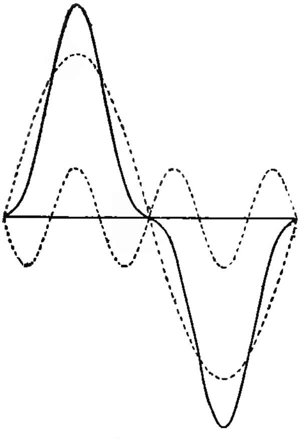

Fig. 37.

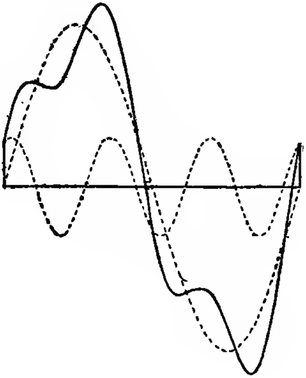

Fig. 38.

is symmetrical about a vertical line bisecting it. Again, if we take an intermediate phase relation, such as that shown in Fig. 38, there is no longer any symmetry about a vertical axis, and the general shape of the resultant curve is intermediate between those of Figs. 36 and 37, the curve being flattened in one part and peaked in another.

It is found that most alternator E.M.F. curves may be represented, with a considerable degree of accuracy, by the expression

$$e = A_1 \sin p\,t + B_1 \cos p\,t + A_3 \sin 3\,p\,t$$
$$+ B_3 \cos 3\,p\,t + A_5 \sin 5\,p\,t + B_5 \cos 5\,p\,t\,. \quad . \quad (26)$$

We shall therefore assume that all the curves with which we have to deal may be resolved into the above harmonic terms. The problem, then, of analysing the curve harmonically is reduced to the determination of the coefficients, A_1, B_1, A_3, B_3, and A_5, B_5. This may be effected as follows:

Taking the zero value of the E.M.F. as our starting point—*i.e.*, as the point at which $t=0$—let us divide a *half-period* into six equal parts. The values of $p\,t$ corresponding to the first six points of division, including the starting point, are

$$0,\ \frac{\pi}{6},\ \frac{\pi}{3},\ \frac{\pi}{2},\ \frac{2\,\pi}{3},\ \frac{5\,\pi}{6}\ \text{radians},$$

or $$0°,\ 30°,\ 60°,\ 90°,\ 120°,\ 150°.$$

From the E.M.F. curve we can read off the corresponding instantaneous values of the E.M.F. Let these be

$$0,\ e_1,\ e_2,\ e_3,\ e_4,\ e_5.$$

Then, remembering that $\sin 30^\circ = \frac{1}{2} = \cos 60^\circ$, and $\cos 30^\circ = \frac{\sqrt{3}}{2} = \sin 60^\circ$, we have, by equation (26),

$$B_1 + B_3 + B_5 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

$$\tfrac{1}{2} A_1 + \frac{\sqrt{3}}{2} B_1 + A_3 + \tfrac{1}{2} A_5 - \frac{\sqrt{3}}{2} B_5 = e_1 \quad . \quad (b)$$

$$\frac{\sqrt{3}}{2} A_1 + \tfrac{1}{2} B_1 - B_3 - \frac{\sqrt{3}}{2} A_5 + \tfrac{1}{2} B_5 = e_2 \quad . \quad (c)$$

$$A_1 - A_3 + A_5 = e_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad (d)$$

$$\frac{\sqrt{3}}{2} A_1 - \tfrac{1}{2} B_1 + B_3 - \frac{\sqrt{3}}{2} A_5 - \tfrac{1}{2} B_5 = e_4 \quad . \quad (e)$$

$$\tfrac{1}{2} A_1 - \frac{\sqrt{3}}{2} B_1 + A_3 + \tfrac{1}{2} A_5 + \frac{\sqrt{3}}{2} B_5 = e_5 \quad . \quad (f)$$

We thus obtain six equations, and these enable us to determine the six unknown coefficients A_1, B_1, etc.

Adding (*b*) and (*f*), and (*c*) and (*e*), we get

$$A_1 + 2 A_3 + A_5 = e_1 + e_5$$

$$\sqrt{3} A_1 - \sqrt{3} A_5 = e_2 + e_4$$

Also, by (*d*), $2 A_1 - 2 A_3 + 2 A_5 = 2 e_3$.

Adding the first and last of this set of equations, we obtain

$$3 A_1 + 3 A_5 = e_1 + e_5 + 2 e_3,$$

which when divided by $\sqrt{3}$ and added to the second equation above gives

$$2 \sqrt{3} A_1 = e_2 + e_4 + \frac{1}{\sqrt{3}} (e_1 + 2 e_3 + e_5),$$

so that $$A_1 = \tfrac{1}{6} \{ e_1 + 2 e_3 + e_5 + \sqrt{3} (e_2 + e_4) \}.$$

From the other equations, we easily obtain

$$A_5 = \tfrac{1}{6}\{e_1 + 2e_3 + e_5 - \sqrt{3}(e_2 + e_4)\},$$

and $$A_3 = \tfrac{1}{3}(e_1 - e_3 + e_5).$$

Next, subtracting (f) from (b), and (e) from (c), we get

$$\sqrt{3}\,B_1 - \sqrt{3}\,B_5 = e_1 - e_5,$$
$$B_1 = 2B_3 + B_5 = e_2 - e_4.$$

Using these two equations, along with (a), we find

$$B_1 = \tfrac{1}{6}\{e_2 - e_4 + \sqrt{3}(e_4 - e_5)\},$$
$$B_5 = \tfrac{1}{6}\{e_2 - e_4 - \sqrt{3}(e_1 - e_5)\},$$
$$B_3 = \tfrac{1}{3}(e_4 - e_2).$$

As an example, we shall take the periodic function represented in Fig. 32. Dividing the half-period into six equal parts, we find the following values of e:

$$e_1 = 9{\cdot}7\,;\quad e_2 = 15{\cdot}06\,;\quad e_3 = 9\,;\quad e_4 = -2{\cdot}95\,;\quad e_5 = -{\cdot}7.$$

Substituting these values in the expressions just obtained for A_1, B_1, etc., we find

$$A_1 = 8\,;\quad A_3 = 0\,;\quad A_5 = 1\,;\quad B_1 = 6\,;\quad B_3 = -6\,;\quad B_5 = 0.$$

Hence the function is capable of being exhibited as the sum of the following simple harmonic terms:

$$8\sin pt + 6\cos pt - 6\cos 3pt + \sin 5pt.$$

The first two terms, which represent two simple harmonic functions of the same period, may be combined into a single term (Art. 19), so that the given function may be written in the form

$$10\sin(pt + \theta) - 6\cos 3pt + \sin 5pt,$$

where $\tan \theta = \frac{3}{4}$. These three harmonic components are shown dotted in Fig. 32.*

46. We shall next show how to apply the above method of harmonic analysis to the determination of the shape of the *current* curve when the E.M.F. curve is given. This method will be best understood by working out a numerical example.

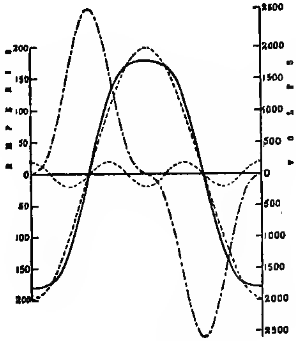

Fig. 39.

In Fig. 39 let the chain-dotted curve be the given E.M.F. curve†. Dividing the half-period into six equal parts we find the following values of the E.M.F. corresponding to the first five points of division—

$$e_1 = 4{,}000\,;\ e_2 = 1{,}730\,;\ e_3 = 2{,}600\,;\ e_4 = 1{,}730\,;\ e_5 = 400.$$

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* It may be as well to point out that the curve of Fig. 32 is not the E.M.F. curve of any alternator; it has been chosen merely to illustrate the method of harmonic analysis given above. Some typical alternator E.M.F. curves are shown in Figs. 31, 33, and 39.

† See Fig. 37, which shows the component harmonics of this curve.

Substituting these values in the expressions given above for A_1, B_1, etc., we find

$$A_1 = 2{,}000\,;\ B_1 = 0\,;\ A_3 = -600\,;\ B_3 = 0\,;\ A_5 = 0\,;\ B_5 = 0,$$

so that the E.M.F. is represented by

$$e = 2{,}000 \sin p\,t - 600 \sin 3\,p\,t.$$

Now the E.M.F.'s which act in any circuit are superposable—*i.e.*, each E.M.F. produces its own current just as if all the other E.M.F.'s were absent. The total resultant current in the circuit will be the algebraical sum of the currents due to the various E.M.F.'s in the circuit. Applying this principle to the case before us, we may determine the currents corresponding to the two simple harmonic components of the E.M.F., and then obtain the actual current by superposing these two.

We shall suppose, in the first place, that the circuit into which the alternating E.M.F. is introduced has negligible inductance, and that its resistance is 10. The impedance of the circuit will in this case be simply equal to its resistance, and the angle of lag will be nothing. Hence, the current in the circuit will be

$$200 \sin p\,t - 60 \sin 3\,p\,t,$$

and since the ratio of the amplitudes, and the phases, of the two simple harmonic components of the current are precisely the same as those of the E.M.F. components, it follows that the *shape* of the current curve will be precisely the same as that of the E.M.F. curve. The chain-dotted curve may, in fact, be taken to represent, to a different scale, the current. This result will obviously apply to any number of simple harmonic components, so that *when the inductance of a circuit is negligible, the current wave is of the same shape as the applied E.M.F. wave.*

We shall, in the second place, consider the other extreme case—that, namely, in which the *resistance* is negligibly small and the inductance considerable. Let the periodicity be 100 ∿ per second (Art. 39), and let the inductance of the circuit be ·0159 henry (Art. 32). The impedance will now be simply equal to the reactance, and the angle of lag will be $\frac{\pi}{2}$ — *i.e.*, a right angle (Art. 40). It must be remembered, however, that the reactance is a function of the periodicity. Hence, since $p = 2\pi n = 2\pi \times 100$, the reactance for the first harmonic term, 2,000 sin $p\,t$, is $p\,\mathrm{L} = 200\,\pi \times \cdot 0159 = 10$; and the reactance for the second harmonic term, $-$ 600 sin $3\,p\,t$, is $3\,p\,\mathrm{L} = 30$. The *currents* corresponding to these two terms will therefore be

$$\frac{2{,}000}{10}\sin\left(p\,t - \frac{\pi}{2}\right) = 200 \sin\left(p\,t - \frac{\pi}{2}\right)$$

and $$-\frac{600}{30}\sin\left(3\,p\,t - \frac{\pi}{2}\right) = -20\sin\left(3\,p\,t - \frac{\pi}{2}\right)$$

respectively. These two harmonic currents are shown dotted in Fig. 39. The actual current in the circuit is obtained by superposing the two harmonic components, and is

$$200 \sin\left(p\,t - \frac{\pi}{2}\right) - 20 \sin\left(3\,p\,t - \frac{\pi}{2}\right).$$

This current is the full-line curve of Fig. 39, and the student cannot fail to be at once struck by the widely different shapes of the E.M.F. and current waves in this case: the one is a *peaked* wave, and the other a *flat-topped* wave. Remarkable as this result may appear at first sight, it might have been predicted from purely physical considerations. For since the resistance of the circuit is negligible, we must at every instant have

$$\text{applied E.M.F.} = \mathrm{L}\, i' \text{ (Art. 33)},$$

where i' denotes the rate of change of the current at the instant under consideration. It follows at once that if the applied E.M.F. has a very small value at any instant, the value of i' must be correspondingly small. Now, a reference to Fig. 39 shows that over a certain portion of the E.M.F. wave—viz., in the neighbourhood of its zero points—the instantaneous values of the E.M.F. are extremely small; hence the corresponding values of i' must also be very small. In other words, the current varies very slowly during that time, and therefore its graph will be almost parallel to the axis of time. This is precisely what the curves in Fig. 39 show: the flat portion of the current wave corresponds to the hollow portion of the E.M.F. wave, over which the values of the E.M.F. are very small.

Another point which it is worth while to refer to in this connection is the following: If we compare the ratios of the *amplitudes* of the two component harmonics corresponding to the E.M.F. and current waves respectively, we find that the second harmonic term is relatively much greater in the E.M.F. than in the current wave—*i.e.*, the E.M.F. wave departs more widely from a pure sine wave than does the current wave. Hence one of the effects of inductance is to *smooth down the current wave** so that it approximates more closely to a sine wave than does the E.M.F. wave.

47. Since we suppose the circuit to be one of constant permeability, the current wave in Fig. 39 will also represent to a different scale the wave of magnetic flux. We thus see that a *peaked* E.M.F. wave gives rise to a *flat-topped* wave of magnetic flux in a circuit whose reactance is great compared with its resistance. If the student will draw the current wave corresponding to a flat-topped E.M.F. wave, he will find that the former is peaked. We thus arrive at the following result:

In a circuit whose resistance is small compared with its reactance a peaked wave of impressed E.M.F. gives rise to a flat-topped wave of magnetic flux, and vice versâ.

This result will also hold for a circuit containing iron, since the E.M.F. induced in any circuit is always measured by the rate of change of the magnetic flux through the circuit, quite independently of the medium in which the flux may be produced. The difference between the two cases lies in the shape of the *current* wave, which in a circuit of constant permeability is similar in shape to the magnetic flux wave, but in a circuit containing iron differs considerably from it.

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* *Not*, be it noted, the induced E.M.F. wave (which in this case is equal and opposite in phase to the *peaked* applied E.M.F. wave). Failure to discriminate between the effects produced by inductance on the *current* and *induced E.M.F.* waves respectively has led to the remarkable fallacy, upheld by high authorities, regarding the "filtering" or "harmonic flywheel" effect of a transformer, or series of transformers, on an E.M.F. wave of irregular shape. It is only quite recently that Dr. Fleming has exploded this fallacy! Inductance *does* have a *smoothing* effect on the current wave, but *not* on the *induced E.M.F.* wave.

Notes on Chapter V.

The student must bear in mind that the method of analysing E.M.F. and current curves explained in the present chapter is *not* a general method, and is based on the assumptions: (*a*) that there are only three harmonic terms in the periodic function; (*b*) that all these harmonic terms are of an *odd* order. The degree of accuracy which may be attained by the use of this method will depend on the accuracy with which the instantaneous values of the function at the points considered may be determined. It is in all cases advisable to reconstruct the curve from the harmonic components obtained, and see in how far it coincides with the original curve; by this means, any errors will be rendered apparent, and the nature of the corrections to be applied may be inferred from a comparison of the two curves. For a general method of analysing periodic curves into their simple harmonic components, the student may consult a paper by Mr. E. Basil Wedmore, entitled "A Simple Method of Analysing Periodic Curves," published in the *Journal of the Institution of Electrical Engineers*, Vol. XXV., page 224. The same volume (page 474) contains an interesting paper by Messrs. Beeton, Taylor, and Barr, in which the effects of different wave forms on the efficiency of transformers are considered, and in which references to other papers on this subject are also given.

EXERCISES.

1. Plot the periodic function represented by the three terms:

$$100 \sin p\,t + 20 \sin\left(3p\,t - \frac{\pi}{3}\right) - 10 \sin\left(5\,p\,t + \frac{\pi}{4}\right):$$

2. Plot the periodic function $50 \sin p\,t + 20 \sin\left(3\,p t + \frac{\pi}{3}\right)$.

3. The half-period included between two zero values of a periodic function known to contain not more than three simple harmonic components of odd order is divided into six parts, and the values of the function at the five points of division included between the two zero values are found to be 2, 8·66, 13, 8·66, and 2 respectively. Find the harmonic components of the function.

CHAPTER VI.

48. The shape of the E.M.F. curve of an alternator may be determined experimentally by the use of an "instantaneous" contact maker, first introduced by Joubert. The action of this device will be best understood by reference to Fig. 40, which shows the arrangement of apparatus required for determining the shape of the E.M.F. curve. T_1 and T_2 are the two terminals of the alternator. E D is an ebonite disc (10in. or 12in. in diameter) which is mounted on the alternator shaft, and which has a thin strip of brass, S, let into its circumference. This strip of brass is connected by means of a wire, W, to an insulated collar, I C, also mounted on the shaft of the alternator. Two brushes, B_1 and B_2, rest, the one on the circumference of the ebonite disc, the other on the insulated collar. The tip of the brush B_1 must be very narrow,

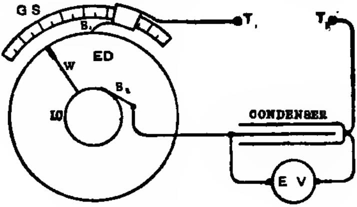

FIG. 40.

so that as the disc, E D, revolves contact is established between B_1 and S once in every revolution for a very short period of time only. The brush B_1 slides on a graduated scale, G S. By varying the position of the brush B_1 we alter the instant at which contact is made with S. The brush B_1 is in connection with one of the alternator terminals, T_1, while B_2 is connected to one terminal of a condenser, the other terminal of which is connected to the alternator terminal T_2. As the alternator revolves, the condenser is connected, in a certain position of the armature depending on the position of the brush B_1 to the terminals of the alternator. As explained above, this contact is of very short duration. By the rapid succession of these brief contacts the condenser becomes charged to, and is maintained at, a P.D. equal to that between the alternator terminals *at the instants when contact is made.* By determining this P.D. for different positions of the brush B_1 we obtain the shape of the E.M.F. curve, equal displacements of B_1 along its scale corresponding to equal intervals of time.

The P.D. between the condenser terminals may be measured by means of an electrometer or an *electrostatic* voltmeter (E V in Fig. 40). The scale of an electrostatic voltmeter is, however, generally so restricted that it would be difficult to obtain accurate readings over the whole range required. This difficulty may be got over by the ingenious device, due to Dr. Fleming, of using an auxiliary E.M.F. in the electrometer circuit. Thus, let us suppose that when the brush B_1 is in a certain position, the corresponding

P.D. is too small to be read on the electrostatic voltmeter. We have merely to connect (Fig. 41) a certain number of cells (secondary cells are very convenient for this purpose) in series with the voltmeter, and so to increase the deflection until it can be read accurately on the voltmeter scale. By subtracting from this reading the E.M.F. of the cells, we obtain the P.D. at the condenser terminals. The same method may be used for reducing the voltmeter reading should the latter prove too high. In this case the cells are connected so that their E.M.F. opposes the P.D. at the condenser terminals, and this P.D. is obtained by *adding* the E.M.F. of the cells to the voltmeter reading.

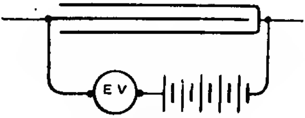

FIG. 41.

Since an electrostatic instrument is used, it might be thought that the condenser is unnecessary. This would be the case if the insulation of the voltmeter were perfect. Owing, however, to the leakage which takes place, it is desirable to give the instrument a large capacity, and this is accomplished by connecting a condenser across its terminals. Even then a certain appreciable amount of leakage may take place, and this will depend on the interval which elapses between two consecutive contacts—*i.e.*, on the speed of the alternator. It is, therefore, always best to calibrate the combination of electrostatic voltmeter and condenser for the particular speed used by applying a known *steady* P.D. to the points T_1 and T_2 of the circuit (Fig. 40), and noting the corresponding reading of the electrostatic voltmeter. If the insulation of the condenser is very high, it will probably be found that there is no appreciable difference between the two readings obtained when the contact is stationary and running respectively. But with a slightly leaky condenser such a difference might exist, and a comparison of the two reading supplies a correcting factor, by which the readings of the electrostatic voltmeter must be multiplied, in order to give the correct values of the instantaneous P.D.

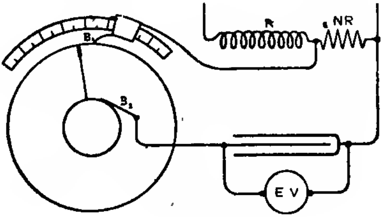

FIG. 42.

49. The method just explained for determining the shape of the E.M.F. wave may also be used for the determination of the *current* wave in an inductive circuit. Fig. 42 shows the arrangement of connections required for this purpose. I R is the inductive circuit in which the current wave is to be determined; N R is a small non-inductive resistance connected in series with the inductive resistance. The readings obtained

with the brush B_1 placed in different positions give us the corresponding instantaneous P.D.'s at the terminals of the non-inductive resistance. Now, since a non-inductive resistance is incapable of offering a counter E.M.F. to a varying current, it follows that at every instant the current through such a resistance is simply equal to the quotient of the P.D. by the resistance. Hence, by dividing the values of the instantaneous P.D. by the resistance we obtain the corresponding values of the current.

In order that the introduction of the non-inductive resistance may not affect the existing conditions of the circuit to any great extent, it is desirable to make this resistance as small as possible. The P.D.'s, however, which will have to be measured in that case will also be small, and an electrostatic voltmeter would not be suitable for this purpose. The best instrument in this case would be an electrometer used heterostatically—*i.e.*, with its needle charged to a high potential and insulated from the quadrants. *Good* electrometers, however, are very expensive instruments, and are not available in every laboratory. A good substitute for an electrometer is provided by the combination of a slide-wire with a galvanometer and an auxiliary source of E.M.F. The arrangement of connections is shown in Fig. 43, where A B represents the slide-wire along which passes a steady current supplied by some constant source of E.M.F. (such as a secondary cell). A reversing switch, R S, is included in this circuit. The point A, being permanently connected to one terminal of the condenser, is maintained at the same potential as that terminal. When the sliding contact of the bridge wire is momentarily depressed at C, it connects the point C to the other terminal of the condenser through the galvanometer, G. If these two points happen to be at the same potential, no current will pass through the galvanometer. The position of C is adjusted until the galvanometer fails to show any deflection. It is obvious that under these conditions the P.D. at the condenser terminals is proportional to the length A C, which is read off on the slide-wire scale.* In order to determine the constant which will convert the scale readings into amperes, a *known* steady current is passed through the non-inductive resistance, N R, the instantaneous contact-maker running at the speed at which it is to be used, and the corresponding scale reading is noted.

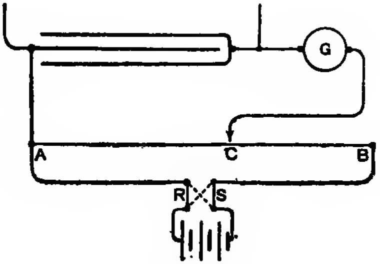

FIG. 43.

By taking corresponding sets of readings for the P.D. and current in any inductive part of a circuit, and plotting the results obtained on a sheet of squared paper, we not only determine the shape of the P.D. and current waves, but also their phase relation, a knowledge of which is frequently of great importance.

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* On the supposition that the resistance of the wire per unit of length is uniform. If this is not the case, the wire must be specially calibrated.

Various modifications of the methods just described for determining the wave-shape have been used by different experimenters, and some important improvements have been introduced into these methods recently. The determination of the wave-shape by the methods given above is a somewhat tedious process, on account of the comparatively long time required to obtain a set of readings. By means of suitably-designed apparatus the wave-shape may be obtained much more quickly.

In some cases it may be necessary to determine the wave-shape when the alternator is not accessible. A special form of apparatus for this purpose has been designed by Dr. Fleming, and another form by Dr. Hicks. In both cases a *synchronous* alternating-current motor is used for driving the instantaneous contact-maker. By a *synchronous* motor is meant one whose speed bears a definite ratio to the periodicity of the alternating current.

Notes on Chapter VI.

Various forms of instantaneous contact-maker have been used by different experimenters; a water-jet and needle contact was devised by Bedell and Ryan, and found to work very satisfactorily. The student will find a description of this device in "A Laboratory Manual of Physics and Applied Chemistry," by Nichols, Vol. II., page 97 (Macmillan and Co.). Prof. Ryan, of the United States, was the first to delineate transformer P.D. and current curves. An account of his researches will be found in Dr. Fleming's "Alternate-Current Transformer," Vol. II., Chap. IV. In Vol. I. of this work (new edition), on pp. 522—525, will be found a description of Dr. Fleming's apparatus, referred to in Art. 49 of the present chapter. M. Blondel (*La Lumière Electrique*, September 12, 1891), and Messrs. Barr, Burnie, and Rodgers (*The Electrician*, Vol. XXXV., page 719) have devised forms of apparatus by means of which the curves may be traced very rapidly.

CHAPTER VII.

50. Since an alternating P.D. or current varies from instant to instant, it is at once evident that if we are to measure such a varying quantity, it becomes necessary to define exactly what we mean when speaking of *one volt of alternating P.D.*, or *one ampere of alternating current.* Since the waves of P.D. and current which we are considering are supposed to be symmetrical with regard to positive and negative half-waves, it follows at once that the mean *algebraical* value of the P.D. or current over any complete number of periods is nothing. We are not, however, concerned with the mean algebraical value, for this gives us absolutely no clue as to the magnitude of the current. In some cases it is important to consider the mean *numerical* value of the current over a complete period, or, which comes to the same thing, its mean value over a positive or negative half-period. In order to determine this value, we may adopt the method used by engineers in working out the mean pressure from an indicator diagram. Thus, let the curve in Fig. 44 represent a positive half-wave of P.D. In order to determine the *mean value* of the P.D., divide the half-period A B into a convenient number of equal parts,* and at the middle point of each

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* The greater the number of divisions, the greater will be the accuracy obtained.

division draw a perpendicular to the axis of time. By taking the sum of the lengths of these perpendiculars and dividing by their number, we obtain the mean value of the P.D. In the example

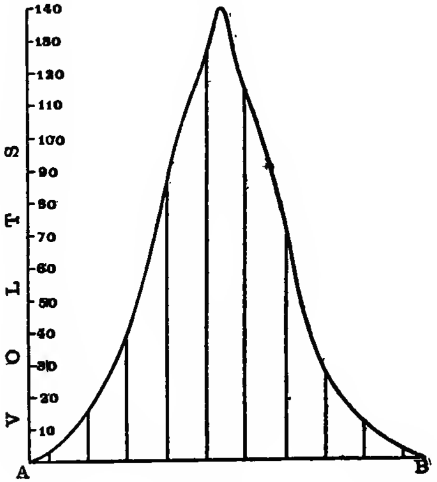

Fig. 44.

under consideration (Fig. 44) we find that the lengths of the perpendiculars are

3, 15, 39·6, 86·4, 126, 115·5, 69·6, 27, 11·4, and 3

respectively. Hence the mean value is

$$\frac{3+15+39{\cdot}6+86{\cdot}4+126+115{\cdot}5+69{\cdot}6+27+11{\cdot}4+3}{10}$$

$$=49{\cdot}65 \text{ volts.}$$

A knowledge of the mean value of a given E.M.F. wave is of importance in the design of alternators, for it is this value which is calculated most readily. As a typical example, let us suppose that an alternator field magnet consists of a frame furnished with polar projections, as shown in Fig. 45, the projections being of north-seeking and south-seeking polarity alternately. Let C C′ be one of the alternator armature coils, of S turns, and let us suppose that in t seconds this coil is carried from its initial position (opposite a north-seeking pole) to the position shown dotted (opposite the neighbouring south-seeking pole). If N stand for the total number of C.G.S. lines which proceed from each pole, the change in the number of lines through each turn of the coil is from + N to − N—*i.e.*, the total change amounts to 2 N. Since the change has taken place in t seconds, the E.M.F., in C.G.S. units, induced in each turn of the coil amounts to $\frac{2\,N}{t}$. Hence the total E.M.F., in volts, induced in the whole coil is

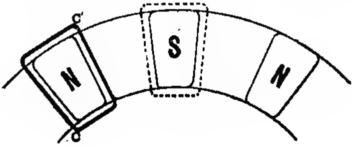

FIG. 45.

$$\frac{2\,N\,S}{t} \times 10^{-8} \text{ volts.}$$

If there are n coils connected in series in the armature winding, then the total *mean* E.M.F. of the alternator is the above expression multiplied by n.

It is thus quite easy to calculate the *mean* E.M.F. developed by a given alternator when the magnetic flux from each pole and the speed are given.

51. In other cases, again, the quantity which it is important to know is the *maximum* value of the E.M.F. In the case of power transmission by means of high-potential alternating currents, the insulation of the line, transformers, etc., must be sufficiently good to withstand the greatest stress to which it may be subjected. But the latter depends solely on the *maximum* value of the E.M.F. employed, so that in this case we are concerned with the *maximum* value, and not the mean value. From this point of view it is evident that, *apart from other considerations*, a *flat-topped* E.M.F. curve is to be preferred to a *peaked* one; for the latter puts a greater strain on the insulation than the former.

52. In most cases, however, it is neither the *mean*, nor yet the *maximum* value of the varying quantity which concerns us most. We now proceed to explain the nature of the quantity which is generally considered in all alternate-current measurement. Take the case of an incandescent lamp supplied with an alternating current. The amount of light given out by the lamp will obviously depend on the *heating effect* of the current. Now as the current varies, its heating effect will also vary from instant to instant. At any moment, however, *the heating effect will be proportional to the square of the corresponding value of the current.* Let in Fig. 46 the full-line curve A D B represent the curve of current. The dotted curve A E B is such that its ordinates are equal to the

squares of the corresponding ordinates of A D B. Thus, the corresponding points D and E are so related that $CE = CD^2$. It follows from this that if A D B represent the variations of the current, A E B will represent the variations in the heating effect produced by the current. In order to determine the *mean*

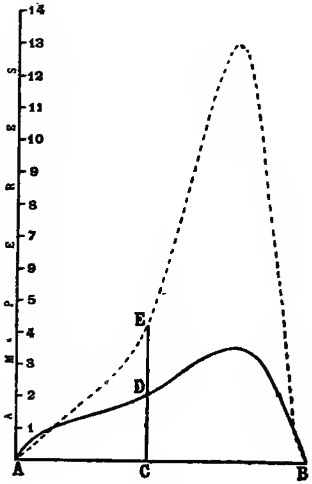

FIG. 46.

heating effect, on which the total amount of light given out by the lamp depends, we must find the mean ordinate of the curve A E B. This is readily done by the method explained above (Art. 50). The mean ordinate of the curve A E B gives us the *mean value of the square* of the current, or, briefly, the *mean square* of the current.

Let us next suppose that the same lamp is supplied with a continuous current which produces *the same heating effect* as the alternating current. In order that this may be the case, it is obvious that the square of the continuous current must equal the mean square of the alternating current. So that for equal heating effects we must have the relation

$$(\text{continuous current})^2 = \text{mean square of alternating current},$$
$$i.e., \text{ continuous current} = \sqrt{\text{mean square of alternating current}}.$$

The inductance of an incandescent lamp is so small that it may be neglected, so that the impedance is practically equal to the resistance of the lamp. From this it follows that the current at any instant is simply equal to the quotient of the P.D. by the resistance. Hence the current wave will have precisely the same shape as, and be in phase with, the P.D. wave, and it follows that if instead of considering currents we consider P.D.'s, then for equal heating effects we must have

$$\text{continuous P.D.} = \sqrt{\text{mean square of alternating P.D.}}$$

Whether, therefore, we consider current or P.D., the important quantity from the point of view of the heating and lighting effects is *the square root of the mean square value.* For the sake of brevity, this latter quantity is generally referred to as the R.M.S. (" root mean square ") value.

53. When we speak of so many volts of an alternating P.D., or so many amperes of an alternating current, we thereby mean their R.M.S. values. We thus have the following definitions:

An alternating P.D. of one volt [alternating current of one ampere] is a P.D. [current] *whose R.M.S. value is unity.*

A unit alternating current might otherwise be defined thus:

An alternating current of one ampere is an alternating current which, in a given conductor maintained at a constant temperature, produces heat at the same rate as that at which it would be produced in the same conductor under the same conditions by a continuous current of one ampere.

The student must be careful to discriminate between the R.M.S. and the *mean* values. Referring to Fig. 46, we find that the mean value of the current is 2·06 (which is the mean value of the ordinate of the curve A D B). The value of the mean square of the current (which is given by the mean ordinate of the dotted curve A E B) is 5·22, so that the R.M.S. value of the current is 2·28.

A knowledge of the relation between the R.M.S. and the mean values of a wave is of very considerable importance in some cases. Thus, let us suppose that we have to design an alternator of a certain type to give 100 volts at a given speed. By the 100 volts is meant the R.M.S. value of the E.M.F. Now the quantity which is directly calculable from the known dimensions and speed of the alternator is *not* the R.M.S. value, but (Art. 50) the mean value. It is thus necessary to know the relation between these two quantities for the given type of alternator before the calculation can be carried out. Dr. Fleming has proposed to give the name "form factor" to the ratio of the R.M.S. value to the mean value. The *form factor* varies in different types of machines from about 1·1 to about 1·4. If the form of the E.M.F. wave is known, then we can at once determine the *form factor.* Thus, we have found that the mean value of the ordinate of the E.M.F. curve, shown in Fig. 44, is (Art. 50) 49·65. In order to find the R.M.S. value, we first of all determine the mean square. This is done by squaring the lengths of the 10 ordinates, which gives us the numbers

9; 225; 1,568; 7,465; 15,876; 13,340; 4,844; 729; 130; and 9,

so that the mean square is 4,419·5, and hence the R.M.S. value is $\sqrt{4,419} = 66{\cdot}5$ volts. The *form factor* for this type of wave is therefore $\frac{66{\cdot}5}{49{\cdot}65} = 1{\cdot}34$.

54. We have next to consider the *measuring instruments* which may be used on alternate-current circuits.

An alternating P.D. may be measured by means of a Cardew or an electrostatic voltmeter. The readings of both these types of instruments are determined by the R.M.S. value of the P.D. In the case of the Cardew voltmeter, the reading of the instrument is determined by the heating effect produced by the current. Owing to the extremely small inductance (Art. 31) of the instrument, the current is in phase with the P.D., and is at any instant equal to the quotient of the P.D. by the resistance of the instrument. From this it follows that an alternating P.D. whose R.M.S. value is E volts will produce a current through the instrument whose R.M.S. value is equal to the continuous current produced by a continuous

P.D. of E volts. But (Art. 52) an alternating current whose R.M.S. value is equal to a certain continuous current produces the same heating effect as this continuous current in a conductor of given resistance. Hence an alternating P.D. whose R.M.S. value is E volts will give the same deflection as a continuous P.D. of E volts, so that *a Cardew voltmeter which has been calibrated with continuous P.D.'s will read correctly when placed on an alternate-current circuit, and the quantity which it measures is the R.M.S. value of the P.D.*

Considering next the case of an electrostatic voltmeter, it will be readily seen that since for a given position of the moving part of the instrument the couple acting on it is proportional to the *square* of the P.D., the deflection produced by an alternating P.D. whose R.M.S. value is E volts will be the same as that produced by a continuous P.D. of E volts; for the mean couple in the first case is the same as the steady couple in the second. Hence an electrostatic voltmeter calibrated with continuous P.D.'s will correctly read alternating P.D.'s.

Electrodynamic and electromagnetic voltmeters are not very suitable for alternate-current circuits, and hence they are not much used on such circuits. Owing to the inductance of the coil of such an instrument, the current which passes through it is only a small fraction of that which would be obtained with a continuous P.D. of the same amount,* so that voltmeters of this kind have to be calibrated by using known *alternating* P.D.'s. Further, since the current passing through the voltmeter depends on the impedance of the coil, and therefore on the periodicity, an instrument calibrated on a circuit of given periodicity will not be correct for another circuit whose periodicity is different. To give the student some idea of the effect produced by the inductance of the voltmeter coil, it may be mentioned that in the case of a voltmeter of the Ayrton and Perry magnifying spring type, a P.D. which, if steady, gives the maximum scale reading, will, if alternating with a periodicity of 100 ∿ per second, produce a deflection which is not even readable on the scale.

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* Unless a large non-inductive resistance be connected in series with the voltmeter coil, so as to make the impedance practically equal to the resistance; in that case, however, the current is also greatly reduced, and the controlling couple must be diminished.

The most satisfactory instruments, from every point of view, for measuring alternating P.D.'s are those of the *electrostatic* type, and these will probably altogether displace instruments of other types.

55. An alternating *current* may be measured by a Siemens electrodynamometer, a Kelvin balance, or an electromagnetic ammeter. The first two instruments named will read the R.M.S. value of the current correctly on an alternate-current circuit if previously calibrated with continuous currents. But an *electromagnetic ammeter* which has been calibrated with continuous currents will *not* read correctly on an alternate-current circuit, and must be specially calibrated with alternating currents of the same periodicity as that of the circuit on which the instrument is to be used.

Notes on Chapter VII.

It must be noted that the second definition of an alternating current of 1 ampere given in Art. 53 is based on the assumption that the resistance of a conductor to an alternating current is the same as its resistance to a continuous current. As will be seen in a later chapter, this is not strictly the case, for the current when alternating is not uniformly distributed over the cross-section of the conductor, the current density being greatest near the surface. The definition is therefore not quite satisfactory, and the first definition is to be preferred.

Exercises.

1. The following are the values of an alternating current at equal time-intervals during a half-period: 0, 1·7, 2·6, 3·2, 3·8, 4·7, 5·7, 6·8, 6·9, 4·5, 0. Plot the current-curve, and find the R.M.S. value of the current.

2. The following are the instantaneous values of an E.M.F. at equal time-intervals during a half-period: 0, 15, 46, 83, 90, 90, 88, 64, 35, 11, 0. Find the form factor of this E.M.F. curve.

CHAPTER VIII.

56. In the preceding articles, we have made no special assumption regarding the shape of the E.M.F. or current wave when discussing the measurement of these quantities. We shall now investigate some special relations which hold for simple sine waves.

We shall in the first place consider the *mean* value of a simple sine function. Since every sine function is capable of being exhibited as the product of a certain constant quantity (the *amplitude* of the sine function, Art. 14) into the sine of an angle which varies at a constant rate with the time, it will be sufficient to determine the mean value of the sine. From this the mean value of the function may be obtained by multiplying by the amplitude (which is the *maximum* value of the function).

Let, in Fig. 47, O P be a line of constant length which, starting from the initial position O A, revolves round O with constant angular velocity. The mean value of the sine of the variable angle A O P over a half-period is obviously the same as its mean value over a quarter-period. Taking, then, the arc A B, which corresponds to a quarter-period, we divide it into a large number n of very small equal parts. If O P $= r$, then the length of the arc A B is $\frac{1}{2} \pi r$, and the length of each division is $\frac{1}{2n} \pi r$. In order to obtain the *mean* value of the sine, we determine its value at the middle point of each division, add the values so obtained, and divide the sum by the number of divisions.

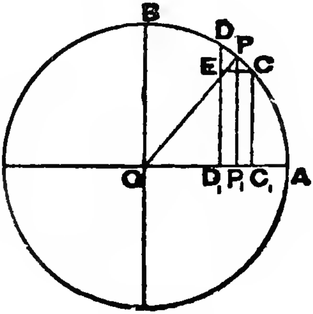

FIG. 47.

Let C D be one of the divisions, P its middle point. Then $CD = \frac{\pi r}{2n}$. Also, since C D is very small (although in the diagram, for the sake of clearness, it is shown fairly large), we may suppose that it is a short straight line. From C, P, and D respectively draw three straight lines at right angles to O A, and from C draw C E parallel to O A. Then, on account of the similarity of the triangles P_1 O P, and E D C, we have

$$\sin P_1 O P = \frac{P P_1}{P O} = \frac{C E}{C D} = \frac{C_1 D_1}{C D}.$$

The same relation will hold for any other division.

If we now take the sum of all the sines so obtained, it is evident that, on account of the length C D being the same for each division, this length will be a common denominator for all the fractions representing the sines. The sum of these fractions will therefore be simply the sum of their numerators, divided by C D. But the sum of the numerators gives us $A\,O = r$, so that the sum of the sines is

$$\frac{r}{C\,D} = \frac{r \times 2\,n}{\pi\,r} = \frac{2\,n}{\pi}.$$

Dividing this sum by the number of divisions, n, we find for the mean value of the sine $\frac{2}{\pi}$. Hence

The mean value of a simple sine function is equal to its maximum value, multiplied by $\frac{2}{\pi}$.

Since $\pi = 3{\cdot}1416$, we thus see that the mean value of a sine function is about ·637 of its maximum value.

57. We have next to consider the R.M.S. value of a sine function. As before, this may be obtained from the *maximum* value by multiplying by the R.M.S. value of the sine. In order to determine the latter, we notice that the relation

$$\sin^2 \theta + \cos^2 \theta = 1$$

holds for all possible values of the angle θ (Art. 8). From this it at once follows that

$$\text{mean value of } (\sin^2 \theta + \cos^2 \theta) = 1,$$

or, $$\text{mean value of } \sin^2 \theta + \text{mean value of } \cos^2 \theta = 1.$$

Now, since the cycle of changes through which the cosine of an angle passes is precisely the same as the cycle through which the sine passes, it follows that the mean value of $\cos^2\theta$ is the same as that of $\sin^2\theta$. We therefore have

$$\text{mean value of } \sin^2\theta = \text{mean value of } \cos^2\theta = \tfrac{1}{2},$$

and the R.M.S. value of the sine or cosine is thus seen to be $\frac{1}{\sqrt{2}}$. Hence

The R.M.S. value of a sine function is equal to its maximum value divided by $\sqrt{2}$.

Since $\sqrt{2} = 1{\cdot}414$, $\frac{1}{\sqrt{2}} = {\cdot}707$, and we see that the R.M.S. value of a sine function is ·707 of its maximum value. Comparing this with the result obtained for the mean value (Art. 56), we see that the R.M.S. value of a sine function exceeds its mean value by an amount which is about 10 per cent. of the former. The *form factor* (Art. 53) for a sine function is $\frac{{\cdot}707}{{\cdot}637} = 1{\cdot}11$.

58. We now pass on to the consideration of the *power* in an alternating-current circuit. At any instant the power is given by the product of the impressed E.M.F. into the current. Our object is to determine the *mean* power over a period. We shall, in the first place, consider the mean value of the power in a circuit whose inductance is negligible. In such a circuit we have at any instant

$$\text{current} = \frac{\text{E.M.F.}}{\text{resistance}},$$

so that

$$\text{R.M.S. value of current} = \frac{\text{R.M.S. value of E.M.F.}}{\text{resistance}}.$$

Again, we have at every instant

$$\text{power} = \text{E.M.F.} \times \text{current} = \frac{(\text{E.M.F})^2}{\text{resistance}}.$$

$$\text{Hence, mean power} = \frac{\text{mean }(\text{E.M.F.})^2}{\text{resistance}}$$

$$= \text{R.M.S. value of E.M.F} \times \frac{\text{R.M.S. value of E.M.F.}}{\text{resistance}},$$

$$= \text{R.M.S. value of E.M.F.} \times \text{R.M.S. value of current.}$$

We thus arrive at the following important result:

In a circuit of negligible inductance, the mean power, in watts, is equal to the product of the E.M.F. in volts into the current in amperes.

The student will notice that this result is true for waves *of any shape*, as we have made no assumption regarding the wave shape in the above reasoning.

We may, therefore, determine the power in any *non-inductive* part of a circuit by measuring the P.D. and current, and taking the product of these two quantities.

59. The above simple rule ceases to hold when the circuit is an inductive one. If the E.M.F. and current waves are known, we can easily determine the mean power by the method already explained for finding the mean value of an E.M.F. or current wave. We have merely to divide a half-period into

a convenient number of equal parts, and determine the power corresponding to the middle point of each division. By adding the values so obtained, and then dividing by the number of divisions, we obtain the mean power in the circuit.

The difficulty which arises in the determination of the power in an inductive circuit is due to the fact that in such a circuit we have, as has been already explained, two E.M.F.'s, the impressed E.M.F. and the induced E.M.F. (Art. 37). During a certain fraction of every half-period, the induced E.M.F. is greater than the impressed E.M.F., so that *the current flows against the impressed E.M.F.* If the student will refer to Fig. 29, he will notice that the impressed E.M.F. is, during a certain fraction of each half-period, of opposite sign to the current. Now in accordance with the principle explained in Art. 27, an E.M.F which is opposed to the current in a circuit becomes a *sink* of energy. During a certain fraction of the half-period, therefore, the source of alternating E.M.F. *absorbs* energy. This means that an alternator working on an inductive load acts as a *generator* during one part of every half-period, and as a *motor* during the remaining part. We thus have a periodic storage and discharge of energy in the circuit, the energy being temporarily stored in the form of a magnetic field surrounding the circuit (Art. 32).

In Fig. 48 the dotted curve is an E.M.F. curve,* and the chain-dotted curve shows the corresponding current curve in a circuit of resistance 3·95 ohms and inductance ·00786 henry, the periodicity being 80 ∼ per second. The full-line curve is the *power* curve, and shows the manner in which the power fluctuates during a period. Between the points A and B, and C and D, the impressed E.M.F. and current are of *opposite* sign, hence the power is

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* The E.M.F. in this example is represented by the two simple harmonic terms $100 \sin p\,t - 30 \sin 3\,p\,t$.

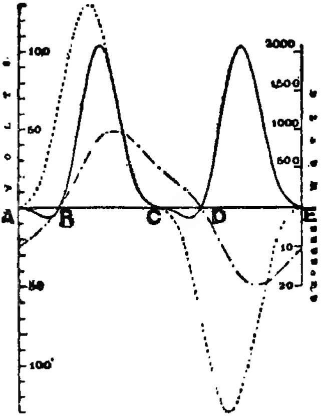

FIG. 48.

negative. Between B and C, and D and E, they are of the *same* sign, hence the power is positive. If at any instant the power is *positive,* then this means that the source of alternating E.M.F. is acting as a *generator*, and supplying energy to the circuit; if, on the other hand, the power is *negative,* then the source of alternating E.M.F. behaves as a *motor,* receiving energy from the circuit.

In order to determine the mean power in the circuit we divide the half-period, A C, into 10 equal parts, and erect perpendiculars to the axis of time at the middle point of each division. The length of these perpendiculars we find to be

$$-20, -100, -100, +300, +1{,}300, +2{,}020, +1{,}800, +900, +250, +20.$$

Their sum is 6,370, and hence the mean power is $\frac{6{,}370}{10} = 637$ watts.

Using the method explained in Art. 53 for determining the R.M.S. value of a periodic function, we find that in the present case the R.M.S. value of the E.M.F. is 73·8 volts, while that of the current is 12·7 amperes. The product of the volts and amperes is therefore $73{\cdot}8 \times 12{\cdot}7 = 937$. But this product, as we have just seen, does *not* represent the true power in the circuit, and is sometimes spoken of as the *apparent watts* in the circuit. The ratio between the *true watts* and the *apparent watts* is termed the *power factor*, and this ratio is a very important quantity from a practical point of view. In the case under consideration, the power factor is $\frac{637}{937} = {\cdot}68$.

60. When the circuit is non-inductive, the apparent watts equal the true watts (Art. 58), so that the power factor is unity. It is in all cases desirable to have the power factor as large as possible, for an increase in the power factor always means an increase in the economy of working. Thus, let us suppose that a certain induc-

tive load is worked by two alternators of the same output, the power factor being ½, and the current supplied by each alternator being the full-load current. Now, although each alternator is supplying its maximum *current*, it is only running at half-load, and so is the engine driving it. Let us contrast this state of affairs with that in which the power factor is unity. Supposing that the P.D. and load are the same in both cases, it follows that only half the current will be required in the second case: Hence a single alternator running at full load will be sufficient, and it is obviously more economical to run one alternator at full load than two alternators at half-load.

There are also other advantages of working with a high power factor. For a given load and P.D., the current varies inversely as the power factor. Hence, the rate at which energy is dissipated by heating of the circuit will vary inversely as the *square* of the power factor. Also, the drop of potential along a given line will vary inversely as the power factor. For both these reasons it is desirable to maintain the power factor as high as possible.

A bank of incandescent lamps, or of closed magnetic circuit transformers *fully loaded,* constitutes an arrangement for which the power factor is practically unity. On the other hand, a load of closed magnetic circuit transformers with open secondaries, of open magnetic circuit transformers (whether loaded or not), and of alternate-current motors, furnish examples of cases in which the power factor may be very small. The power factor may be increased by the use of condensers. This effect will be discussed later on.

61. If the E.M.F. and current in a circuit obey the simple harmonic law, and the angle of lag (Art. 18) is known, then we can at once calculate the power. In order to show how this may be done we shall have to prove the following two auxiliary propositions:

I. *The mean value of the product of two simple harmonic functions which are in phase with each other is equal to half the product of their maxima values.*

Let the two functions be represented by $A_1 \sin p\, t$ and $A_2 \sin p\, t$ respectively. At any time t, their product is equal to $A_1 A_2 \sin^2 p\, t$ (Art. 8). Hence the mean value of this product will be equal to $A_1 A_2$, multiplied by the mean value of $\sin^2 p\, t$. But the latter quantity we have shown (Art. 57) to be equal to $\frac{1}{2}$. The mean value of the product considered is therefore $\frac{1}{2} A_1 A_2$, which establishes the proposition.

II. *The mean value of the product of two simple harmonic functions which are in quadrature with each other* is equal to nothing.*

Let A L C M E (Fig. 49) and R B S D T be two sine functions which are in quadrature, and let the dotted line represent the values of the product of the two functions. Between A and B, and C and D, this product is negative, since the functions are of opposite sign; between B and C, also D and E, the product is positive, since the two functions are of the same sign over these intervals. Consider two points, H and N, which are equidistant from B. From the symmetry of the sine function it follows that H O = N P, and I H = N G. Hence the product of the two sine functions at the point H has precisely the same *numerical* value as at the point K. The same holds for every pair of points equidistant from B. Hence it follows that the positive half-waves of the dotted curve are equal in every respect to the negative half-waves, and, therefore, the mean value of the product considered over any complete number of periods is nothing.

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* Two simple harmonic functions are said to be *in quadrature* with each other when the phase difference between them amounts to $\frac{\pi}{2}$ (one right angle).

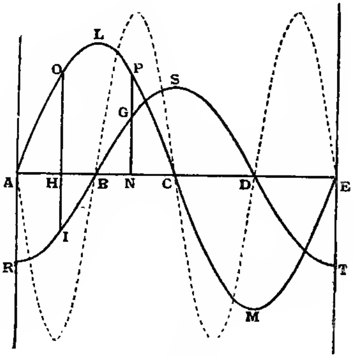

Fig. 49.

We may now apply the above results to the determination of the mean product of two sine functions with any phase difference between them whatever. Let the two functions be represented by the alternating vectors derived from the rotating vectors

O A and O B (Fig. 50), the phase difference between which amounts to θ (Art. 18). Let the rotating vector O B be replaced by the two equivalent rotating vectors O C and O D (Art. 18), the first of which coincides in direction with O A, while the second is at right angles to it. This is equivalent to splitting up the simple sine wave L M N O P into two simple harmonic components (shown dotted in the figure), which are in quadrature with each other. If the sine function derived from the rotating vector O A be represented by O A sin $p\,t$ (Art. 16), then that derived from O B will be represented by O B sin $(p\,t-\theta)$. But, referring to the figure, we see that (Art. 12)

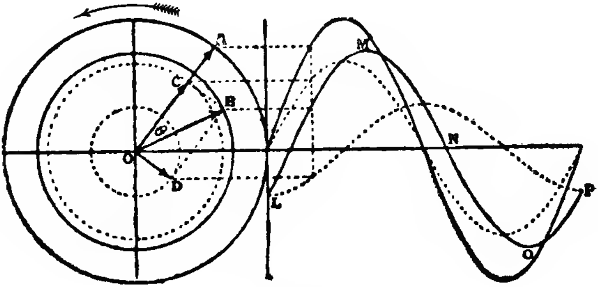

FIG. 50.

$$\text{O B} \sin (p\,t-\theta) = \text{O C} \sin p\,t + \text{O D} \sin \left(p\,t-\frac{\pi}{2}\right).$$

The problem is to find the mean value of

$$\text{O A} \sin p\,t \times \text{O B} \sin (p\,t-\theta),$$

i.e., of

$$\text{O A} \sin p\,t \left\{ \text{O C} \sin p\,t + \text{O D} \sin \left(p\,t-\frac{\pi}{2}\right)\right\},$$

or,

$$\mathrm{O\,A}\sin p\,t \times \mathrm{O\,C}\sin p\,t + \mathrm{O\,A}\sin p\,t \times \mathrm{O\,D}\sin\left(p\,t + \frac{\pi}{2}\right).$$

Applying proposition I. proved above, we find that the mean value of the first term is ½ O A × O C. The mean value of the second term is, by proposition II., nothing. Hence the mean value of the product of the two functions is ½ O A + O C. But O C = O B cos θ (Art. 12), so that the mean value of the product is ½ O A × O B × cos θ. If, now, we suppose that the sine function derived from O A represents the E.M.F. in a circuit, and that derived from O B the current, then we have the following proposition :

If the E.M.F. and current in a circuit follow the simple harmonic law, then the power is equal to half the product of the maxima values of these quantities, multiplied by the cosine of the angle of lag.

62. The component of the current which is in quadrature with the E.M.F. is, as we have seen (Art. 61, Prop. II.), a current which does no work in the circuit; it is concerned with the periodical storage and discharge of energy which go on in the circuit in consequence of its inductance. If in Fig. 49 the curve A L C M E represent the E.M.F. wave, and R B S D T that component of the current wave which is in quadrature with the E.M.F., then the dotted curve will be the power curve corresponding to that component, and the resultant power is nothing. For this reason, the component of the current which is in quadrature with the E.M.F. is sometimes termed the *idle* or *wattless* current.

The other component of the current, being in phase with the E.M.F., is always of the same sign as the E.M.F., hence the power due to this component is always positive. For this reason the component is sometimes referred to as the *load* current.

In Fig. 51 is shown the power curve corresponding to the E.M.F. and current curves of Fig. 50. This power curve may be analysed into two components:

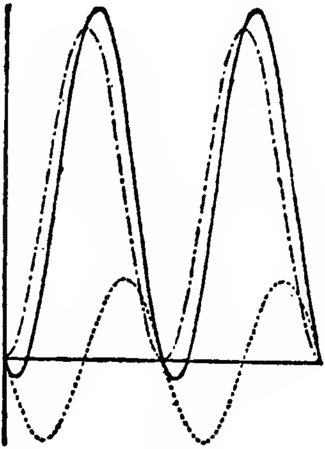

Fig. 51.

the one (shown dotted) corresponding to the *wattless* current, and consisting of equal positive and negative half-waves; the other (shown chain-dotted) corresponding to the *load* current, and containing positive half-waves only.

It is hardly necessary to draw the student's attention to the fact that a *wattless* current could not exist *by* *itself.* In order that this might be the case, it would be necessary to make the angle of lag $\frac{\pi}{2}$, a condition which would require the resistance of the circuit to be equal to nothing (Arts. 38 and 2).

63. If E and I denote the maxima values of the E.M.F. and current, which we suppose to obey the simple harmonic law, and if θ stand for the angle of lag, then, as we have seen (Art. 61), the power is $\frac{1}{2}$ E I cos θ. This cosine of the angle of lag is a quantity which, under different guise, we have already had occasion to consider when dealing with the more general case in which the E.M.F. and current may vary according to any law (Art. 59). For $\frac{1}{2}$ E I $= \frac{E}{\sqrt{2}} \times \frac{I}{\sqrt{2}}$, and by the results established in Art. 57 $\frac{E}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}$ are the R.M.S. values of the E.M.F. and current respectively. Hence $\frac{1}{2}$ E I gives us the *apparent watts* in the circuit—*i.e.*, the product of the volts and amperes (Art. 59); and cos θ is the multiplier which converts the apparent into the true watts. We thus see that cos θ is simply the *power factor* (Art. 59) for the circuit considered.

64. The presence of inductance in a circuit has the effect of displacing the current relatively to the E.M.F., thus giving rise to an idle component of the current, and lowering the power factor. But, like many other things, inductance is not an unmitigated evil, and in some cases is made to serve very useful and important purposes. We shall work out a

numerical example to show how in some cases considerable economy may be effected by pressing inductance into service.

Let a group of 10 incandescent lamps, each of which takes 1 ampere at 60 volts, be required to be run off alternate-current mains at 100 volts, with a periodicity of 100 ∿ per second, the E.M.F. following the simple sine law. The lamps being connected in parallel, their joint resistance will be 6 ohms. The problem, then, is to reduce the P.D. from 100 to 60. This might be done by using a simple resistance of 4 ohms. The objection, however, to this method (which is the only one available in the case of continuous currents[*]) is that the resistance would absorb fully $4 \times 10^2 = 400$ watts, the lamps themselves requiring only $60 \times 10 = 600$ watts. Obviously, such a method would be very wasteful. There is, however, another method which is available with an alternating-current supply. This consists in using a *choking coil* to reduce the P.D. A *choking coil* is a coil (generally provided with an iron core) whose inductance is great compared with its resistance, and which is used in the same way in alternate-current working as a resistance is in the case of continuous currents. As an approximation, we shall suppose that the resistance of our choking coil is negligibly small, and that its inductance is L. The impedance of the coil will be equal (approximately) to the reactance p L (Art. 38). The resistance of the lamps being 6 ohms, the impedance of the whole circuit A B (Fig. 52) across the mains will be $\sqrt{6^2 + p^2 L^2}$. Now the current required is 10 amperes, and the P.D. is 100. Hence, we must have

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[*] Unless, indeed, we use a continuous-current transformer.

$$10 = \frac{100}{\sqrt{36 + p^2 L^2}},$$

from which we find $pL = 8$, so that $L = \frac{8}{2\pi \times 100}$ $= {\cdot}0127$ henry. A coil having this inductance will therefore reduce the P.D. so that the lamps will work at 60 volts.

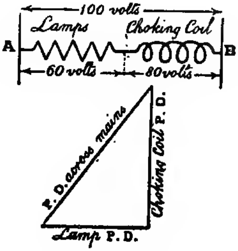

FIG. 52.

65. The obvious advantage of a choking coil over a simple resistance lies in the fact that the former absorbs very little power; if we could entirely get rid of resistance* the coil would absorb absolutely no power. In practice, choking coils are largely used with arc lamps for parallel working.

66. There is one point in connection with the problem we have just considered to which it may be well to draw the student's attention. If a non-

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* And *hysteresis* in the iron core, if one is used.

inductive resistance is used to reduce the P.D., then the P.D. at the terminals of this resistance will be $4 \times 10 = 40$ volts, and the P.D. across the mains is the sum of the P.D.'s across the lamps and resistance respectively. But if a choking coil is used, then, calling x the unknown P.D. at the terminals of the choking coil, and remembering that

P.D. = current × impedance (Art. 38, equation 23),

we have

$$x = 10 \times \mathrm{L}p = 10 \times 8 = 80 \text{ volts.}$$

The P.D. at the terminals of the choking coil is therefore not 40, as the student might at first sight have supposed, but 80. The sum, therefore, of the P.D.'s across the lamps and choking coil respectively does *not* give the P.D. across the mains. The reason for this, at first sight, somewhat puzzling fact lies in the phase difference between the two quantities. In the case where a simple resistance is used the two P.D.'s are both in phase with the current; but the P.D. of the choking coil is in quadrature with the current, so that in this second case the P.D. across the mains is represented by the hypotenuse of a right-angled triangle having its two sides equal to the P.D.'s across the choking coil and lamps respectively, as shown in Fig. 52. In this connection the student would do well to refer back to Arts. 37 and 38.

Notes on Chapter VIII.

The method of determining the power in an inductive circuit explained in Art. 59 has been used by Prof. Ryan (see "The Alternate-Current Transformer," by Dr. Fleming, Vol. II., page 450) and by Dr. Hopkinson (*The Electrician*, Vol. XXIX., page 196). Although capable of giving good results, this method is very laborious. In the next chapter the student will find an account of the methods generally used for the measurement of power in inductive circuits.

In Vol. I. of "The Alternate-Current Transformer" (new edition), pp. 566 *et seq.*, the student will find tables of the *power factors* of different types of transformers when working on open circuit and fully loaded.

EXERCISES.

1. The following figures represent simultaneous values of the P.D. and current of a transformer at instants separated by equal time-intervals:

P.D. ……	+119	+855	+1250	+1420	+1330	+900	+45	−751	−1210
Current…	−·20	−·05	+·07	+·11	+·14	+·19	+·21	+·04	−·08

Plot the curves, and from them find the power curve, and determine the mean power absorbed by the transformer.

2. The P.D. at the terminals of a transformer is 2,400 volts, the current is ·112 ampere, and the power is found to be 230 watts. What is the power factor?

3. In an alternate-current power transmission plant, a drop of 10 volts occurs along the line when the power factor is ·85. If the load remains equal to its former value, but the power factor changes to ·75, what will be the drop of potential along the line, and by how much will the power lost in the line be increased?

4. An E.M.F. represented by $200 \sin 500\,t + 50 \sin \left(1500\,t + \frac{\pi}{4}\right)$ is applied to a circuit of resistance 5 ohms, and inductance ·15 henry. Find the mean power in the circuit.

5. An alternating-current arc lamp takes 15 amperes at 30 volts. If the lamp is to be run off 100-volt mains, what must be the inductance of a choking coil connected in series with it? Treat the lamp as a simple non-inductive resistance, and assume sine law.

CHAPTER IX.

67. The result deduced in Art. 61 for the power in a circuit in which both E.M.F. and current obey the simple harmonic law is not of any importance as furnishing a practical method of measuring the power in an inductive circuit. Even if the simple harmonic law were followed in a given case, we should have to know the angle of lag, and the best method of determining this would, in fact, be to measure the power. The power measurement would thus precede the measurement of the angle of lag. The student must therefore guard against the fallacy that the result of Art. 61 furnishes us with a *practical* method of measuring power in an inductive circuit.

Any method of measuring power which is to be of use in practice must be applicable to E.M.F's and currents of any shape whatever. In the next few articles we shall give an account of two of the most important methods of determining the power in an inductive circuit. A large number of methods have been invented from time to time, but by a process of selection only a few of these have survived to the present day.

68. When the circuit in which the power to be measured is a non-inductive one, the measurement presents no difficulty. For, as was shown in Art. 58,

the true watts are in that case equal to the product of the volts and amperes, the power factor being unity. It is only where we have inductance in the circuit, and where consequently we have to reckon with an unknown power factor, that difficulties occur.

69. The first method which we shall describe for measuring power in an inductive circuit is that known as the *three-voltmeter* method. The arrangement of connections for this method is shown in Fig. 53, where A B denotes the inductive circuit in which the power is to be measured, B C a non-inductive

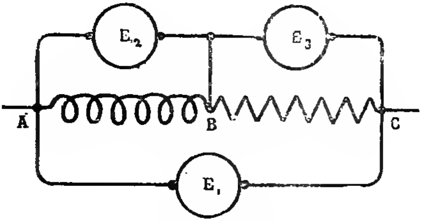

FIG. 53.

resistance of amount r connected in series with A B, and E_1, E_2, E_3 three voltmeters connected as shown. Let v_1, v_2, and v_3 denote the *instantaneous* values of the P.D.'s between A and C, A and B, and B and C respectively. Then we must have, at the instant considered,

$$v_1 = v_2 + v_3 \quad . \quad . \quad . \quad . \quad . \quad (a)$$

Next, the instantaneous power in A B is the instantaneous P.D. between its terminals—*i.e.*, v_2—multiplied by the instantaneous current. The latter, however, is the same as the instantaneous current in B C, which, since B C is a *non-inductive* resistance, is equal to $\frac{v_3}{r}$. We thus have

$$\text{instantaneous power in A B} = \frac{v_2 \cdot v_3}{r}.$$

But from the equation (α) above we find

$$v_1^2 = v_2^2 + 2 v_2 v_3 + v_3^2,$$

whence $$v_2 v_3 = \tfrac{1}{2} (v_1^2 - v_2^2 - v_3^2).$$

Substituting this value for $v_2\ v_3$ in the expression for the instantaneous power, we get

$$\text{instantaneous power in A B} = \frac{1}{2r} (v_1^2 - v_2^2 - v_3^2),$$

and hence

$$\text{mean power in A B} = \frac{1}{2r} (\text{mean } v_1^2 - \text{mean } v_2^2 - \text{mean } v_3^2).$$

Now if V_1, V_2, and V_3 denote the readings of the three voltmeters, we have $V_1^2 = \text{mean } v_1^2$, etc., so that

$$\text{mean power in A B} = \frac{1}{2r} (V_1^2 - V_2^2 - V_3^2).$$

The power is thus given in terms of the readings V_1, V_2, and V_3 of the three voltmeters, and the value r of the non-inductive resistance.

70. In deducing the above expression for the power in the inductive circuit, we have made no supposition regarding the shape of the E.M.F. and current waves. The reasoning is perfectly general, and will hold for any shape of wave.

Instead of using three voltmeters, a single voltmeter may be used for taking the three readings consecutively. But it is always advisable to use three instruments if these are available, as it is important that the three readings should be taken at the same instant.

The non-inductive resistance r should be so chosen that the reading V_3 is as nearly as possible equal to V_2.

On the whole, the three-voltmeter method of measuring power cannot be considered a thoroughly satisfactory one. It is in most cases difficult to get good results with it, and sometimes comparatively trifling errors in the readings may give rise to a serious error in the determination of the power. Again, in order to carry out the measurement under the best possible conditions, the resistance r must be made large enough to absorb more power than that which is taken by the inductive load under test. For, as mentioned above, V_2 and V_3 must be made as nearly equal as possible to secure the greatest accuracy, and since the power factor for r is unity, while that for A B is generally less than 1, it follows that r will absorb more power than A B The introduction of the resistance r, therefore, altogether upsets the existing conditions of the circuit, and in order to restore these we must provide a source of higher E.M.F. (about double that required by A B). It is not always convenient to do this, and hence the three-voltmeter method is necessarily limited in its application.

71. The most generally useful method of measuring power in an inductive circuit is that involving the use of a *wattmeter*. A *wattmeter* is an electrodynamo-

meter provided with two coils—a fixed coil of thick wire and a movable coil of fine wire. The fine-wire coil is connected in series with a large non-inductive resistance, so that the time-constant (Art. 35) of the fine-wire circuit is extremely small, and hence its impedance practically equal to its resistance (Art. 38); the current in the fine-wire circuit will under these conditions be practically in phase with the P.D. across its terminals, the angle of lag being extremely small.

In Fig. 54 A B represents the inductive circuit in which the power is to be determined, B C the thick-wire coil of the wattmeter, E the fine-wire coil, connected in series with the non-inductive resistance N R. The couple acting on the fine-wire coil at any instant is proportional to the product of the currents through the two coils at that instant. Now the current through the fine-wire coil is, as stated above, in phase with the P.D. across A C, and is proportional to this P.D. Again, the current through the thick-wire coil B C is the same as the current through A B. Hence the couple acting on the fine-wire coil is proportional, at a given instant, to the instantaneous product of the P D across A C into the current through A C, and therefore the reading of the instrument,

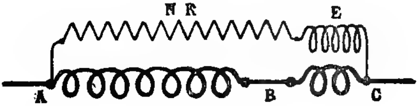

FIG. 54.

which depends on the mean value of the couple, will be proportional to the mean power. By multiplying the wattmeter reading by a certain constant, we at once obtain the power in A C.

Since the resistance of the thick-wire coil of the wattmeter is very low, the introduction of it into the circuit does not materially alter the conditions already existing in the circuit; in this respect, the wattmeter method of measuring power has a great advantage over the three-voltmeter one. The amount of power absorbed by the thick-wire coil is, it must be noted, included in the measurement. An allowance may be made for this, if necessary, by subtracting from the total power measured the product of the resistance of the thick-wire coil into the square of the current (which may be measured by an ammeter included in the circuit, outside A C*).

The constant of the wattmeter may be determined by using, in place of A B, a known non-inductive resistance, and measuring the current through it and the P.D. across its terminals. The product of these two quantities will give the power (Art. 58), and if the corresponding reading of the wattmeter be noted the constant of the instrument is at once determined.

72. At one time the wattmeter was not considered a satisfactory instrument. Readings obtained with it were found to be unreliable, and the instrument came to be looked upon with suspicion. The reason of this, however, lay in the faulty construction of the instru-

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* The ammeter might also be placed in the part of the circuit included between A and C; in this case, however, the amonnt of power absorbed by the ammeter must be added to that taken up by the thick-wire coil of the wattmeter.

ment, and a good modern wattmeter furnishes the best method of measuring the power in an inductive circuit. There are, in particular, two important points which should be carefully attended to in constructing a wattmeter. The first is that the time-constant of the fine-wire circuit should be negligibly small, so that the current through it is for all practical purposes in phase with the P.D. at its terminals. The second point is that the supports of the coils, etc., should be constructed entirely of insulating material. If the instrument is provided with a case, the case should also be of insulating material. The presence of any large masses of metal in the neighbourhood of the wattmeter seriously affects its readings, owing to the currents which are induced in these masses by the alternating magnetic field due to the fixed wattmeter coil. These eddy currents react on the wattmeter and alter its reading. The student will understand this action better when we come to deal with electro-magnetic repulsion phenomena, which will form the subject of the next few articles.

Notes on Chapter IX.

The accurate measurement of power in an inductive circuit is a problem of very great importance, for on it depends the determination of the efficiency of alternate-current transformers, motors, etc. A number of interesting questions regarding the behaviour of a transformer under varying load remained for a long time unsolved, on account of the difficulties which were formerly experienced in measuring the power given to the transformer. The wattmeter method was at that time discredited, and the ingenuity of many experimenters was devoted to devising other methods. One of the earliest of these is the electrometer method, which was independently proposed by Profs. Ayrton and Fitzgerald and M. Potier in 1881. An account of this method will be found in the *Transactions of the*

Royal Society of London, Vol. 182, in the paper on "Quadrant Electrometers." Mr. Blakesley devised a method known as the *split-dynamometer* method. The three-voltmeter method is due to Messrs. Swinburne, Ayrton, and Sumpner. A modification of this, known as the *three-ammeter* method, was subsequently proposed by Dr. Fleming (*The Electrician*, Vol. XXVII., p. 9). A comparative account of some of the methods as applied to the determination of the efficiency of transformers will be found in Dr. Fleming's paper on "Experimental Researches on Alternate-Current Transformers" (*Journal of the Institution of Electrical Engineers*, Vol. XXI., p. 594).

In employing any method involving the use of a non-inductive resistance, it is always advisable to make sure that the resistance is really non-inductive. This may be readily ascertained by sending an alternating current through it, and measuring the P.D. across its terminals. If the quotient of the P.D. by the current does not differ appreciably from the resistance as determined by using continuous currents, we may infer that the resistance is truly non-inductive.

CHAPTER X.

73. The effect of inductance in an alternating-current circuit is a twofold one. It diminishes the current in the circuit by increasing the impedance, and it causes the current to lag behind the impressed E.M.F. as regards phase. This lag of the current behind the impressed E.M.F. gives rise to many remarkable effects. Some of these, which have been more particularly studied by Prof. Elihu Thomson, we shall now proceed to explain. The interest which attaches to these phenomena is not a purely theoretical one, for the principles underlying them have received a practical application in the construction of a number of alternate-current instruments.

74. Consider a conducting mass placed in a magnetic field, and let the field be made to undergo a change. During the time that the field is changing, E.M.F.'s are induced in the conducting mass, and give rise to currents in it. The direction of these currents is, by Lenz's law (Art. 24), such that they tend to oppose the change in the magnetic field which gives rise to them. Hence the conducting mass will tend to move (either by a motion of translation, or by a motion of rotation, or by a combination of both) so as to prevent the change in the magnetic flux through it.

As an example, suppose that a short cylinder of metal L M is placed in the magnetic field shown in Fig. 55. For the sake of simplicity, we shall suppose that the cylinder is supported by a very long silk fibre attached to the point O. This will allow the cylinder considerable freedom of motion. Let the magnetic field be now weakened. The currents induced in the cylinder will have the direction indicated by the arrow A B when looked at from the side L of the cylinder.

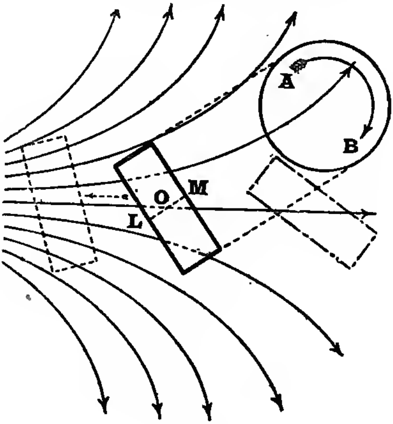

FIG. 55.

These currents will tend to move the cylinder so as to maintain the magnetic flux through it unaltered—*i.e.*, so as to produce by the motion an increase in the number of lines while the field is diminishing. Now, an increase in the number of lines passing through the cylinder may be brought about in two ways: (1) by a bodily movement of the cylinder into a stronger part of the field—hence the cylinder will tend to move bodily in the direction of the dotted arrow; (2) by a rotation of the cylinder about a vertical axis in such a direction that its plane faces tend to become perpendicular to the direction of the field. In other words, the cylinder will be acted on by a force and a couple; the first will produce a motion of translation, the second a motion of rotation, and the cylinder will be deflected and rotated into some such position as that shown dotted in the figure.

If we suppose that the field is increased instead of being diminished, then all the actions we have just considered will be reversed, and the cylinder will tend to swing into the position shown chain-dotted in the figure.

75. Let us next suppose that the magnetic field in which the cylinder is placed is made to alternate between two equal and opposite values. This effect might be produced by the pole of an alternate-current electromagnet held near the cylinder. For the sake of simplicity, we shall suppose that the magnetic flux through the cylinder varies according to the simple harmonic law. Let the curve A B C D E (Fig. 56) represent the variations in the intensity of the magnetic field. The rate of change of the magnetic field will be represented by another sine curve (shown dotted in Fig. 56) which is in quadrature with the first curve (Art. 22). Now since the E.M.F. induced in any circuit inside the cylinder is measured by the rate of change of the magnetic flux through it (Art. 23), it follows that all such E.M.F.'s will be represented by curves such as the dotted curve in the upper part of Fig. 56. Again, if the circuits along which the

currents flow inside the cylinder were devoid of inductance, the currents in them would be in phase with their respective E.M.F.'s. Hence the currents would also be represented by curves such as the dotted curve in Fig. 56. The force or couple due to any one of these currents would at any instant be proportional to the product of current and field intensity at that instant. But the mean value of two sine functions which are in quadrature with each other is zero (Art. 61). Hence there would be no resultant action on the cylinder, the successive impulses which it receives being represented by equal positive and

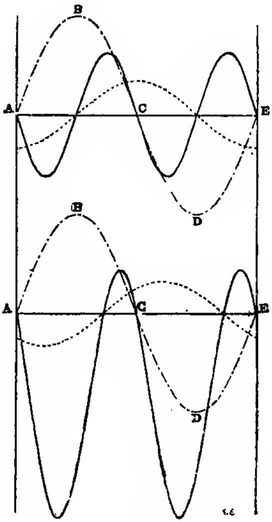

FIG. 56.

negative half-waves, as shown by the full-line curve in the upper part of Fig. 56.

We know, however, that the circuits in which the induced currents flow in the cylinder have a certain amount of inductance. Owing to this fact, the currents, instead of being in phase with their respective E.M.F.'s, lag behind them. The dotted curve in the lower part of Fig. 56 is supposed to represent one of the currents, and it will be noticed that this curve lags behind the dotted curve in the upper part of the figure. The result is that now the successive impulses received by the cylinder are no longer balanced; the negative impulses considerably exceed the positive ones, as shown by the full-line curve in the lower part of the figure. On the whole, therefore, the cylinder will experience a negative force and negative couple, and it will tend to take up permanently some such position as that shown chain-dotted in Fig. 55. In other words, there will be an apparent *repulsion* of the cylinder by the pole of the alternating electromagnet, and the cylinder will tend to turn round so as to make the magnetic flux through it a minimum.

76. Instead of dealing with a large conducting mass placed in a variable field, let us next suppose that we have a simple rectangular frame of wire, A B C D (Fig. 57), suspended as shown in a field which is uniform,* and which varies as time goes on according to the simple harmonic law. In accordance with the principles explained above, the frame will tend to move

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* *i.e.*, which has the same value from point to point of the space considered. *Uniform* refers to *space*, *constant* to *time*. A field may be *uniform* and yet not *constant*.

so as to reduce the magnetic flux through it to a minimum. Now since the field is uniform, a pure motion of translation would not reduce the magnetic flux through the circuit. The frame therefore simply tends to turn round so that its plane may become parallel to the field. It experiences a couple due to the forces acting on the sides A B and D C. Let each

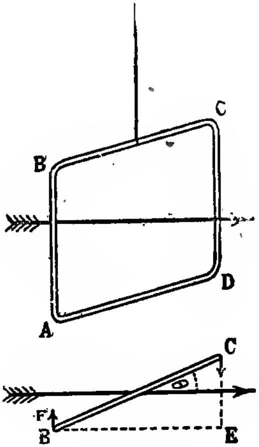

Fig. 57.

of these forces be denoted by F, and let θ be the angle which the plane of the circuit makes with the direction of the field. We shall investigate for what value of the angle θ the couple becomes a maximum. Referring to the diagram (Fig. 57), and denoting by c the couple acting on the frame, we have

$$c = \mathrm{F} \times \mathrm{B\,E}.$$

Now F is, for a given field intensity, proportional to the current, and therefore to the E.M.F. induced in the frame; this E.M.F. is measured by the rate of change of the magnetic flux, and is therefore proportional to the maximum value of the flux. The latter, again, is proportional to E C; so that

$$c \propto \mathrm{E\,C} \times \mathrm{B\,E}$$

But E C = B C sin θ, and B E = B C cos θ (Art. 6). Hence, since B C is a constant,

$$c \propto \sin\theta \times \cos\theta,$$

or,

$$c \propto \sin 2\,\theta,$$

since by Art. 8, equation (8), $\sin 2\,\theta = 2 \sin\theta \cos\theta$. The greatest value of the couple will therefore be obtained by placing the frame in such a position that $\sin 2\,\theta$ has the greatest possible value. But the greatest value of the sine of an angle is 1; and the corresponding value of the angle is $\frac{\pi}{2}$ (a right angle). Hence θ must be made equal to 45deg. if the couple experienced by the frame is to be as great as possible.

77. A galvanometer for measuring very small alternating currents has been constructed by Dr. Fleming, in which the above principles are made use of. It consists of a coil of wire, inside which is suspended a small disc of copper whose plane makes an angle of 45deg. with the axis of the coil. The couple acting on the disc is proportional to the product of the current induced in it into the field intensity. But the induced current is itself proportional to the field intensity. Hence the couple acting on the disc varies

as the *square* of the field intensity—*i.e.*, as the square of the current through the galvanometer coil. If we suppose that the deflections of the disc are small, so that the couple remains practically unaltered by the displacement of the disc, then we see that the deflections of such a galvanometer are proportional to the *square of the current.*

A number of other instruments have been devised whose action depends on the principles explained above.

78. Before dismissing this subject, we shall consider the effect of phase difference between armature E.M.F. and current on the field of an alternator.

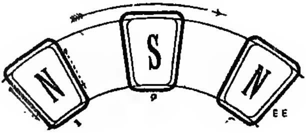

FIG. 58.

Let, in Fig. 58, N S N denote the polar projections of an alternator, and let 1, 2, and 3 be three positions of an armature coil. The E.M.F. passes through its zero value when the coil occupies the positions shown. Between the positions 1 and 2 the direction of the E.M.F. is that shown by the full-line arrow; between 2 and 3, that shown by the dotted arrow. The positions of the coil indicated in the figure are those in which the armature current can most readily affect the field, since the coils in these positions are facing the polar projections. If the armature current is in phase with the E.M.F., then, on the whole, it exerts no strengthening or weakening effect on the field; for in the positions of the coils calculated to produce the greatest reaction on the field, the current passes through its zero value; and in other positions of the coils such weakening effects as are produced when a coil is approaching a pole are balanced by equal strengthening effects when the coil is receding from the pole.

Suppose, however, that the alternator is working on an inductive load, and that consequently the current lags behind the E.M.F. In this case, when the coil is in the position 1, there is a current flowing in it, whose direction is indicated by the dotted arrow. The coil will therefore exert a demagnetising effect in this position, and this effect will increase with the lag of the current. We thus see that a *lagging* current *weakens* the field of an alternator.

Again, let us suppose that the current is in advance of the E.M.F. as regards phase. Then the action will obviously be reversed, and hence a *leading* current *strengthens* the field of an alternator.

Notes on Chapter X.

A number of interesting experiments on electromagnetic repulsion will be found described in Dr. Fleming's "Alternate-Current Transformer," Vol I. (new edition), p. 307. The principles explained in the present chapter have been practically applied in the construction of alternate-current meters, the meters of Messrs. Borel, Wright, and Ferranti, and that of Shallenberger, being two well-known examples.

Prof. Elihu Thomson has found that if a copper disc whose diameter is less than that of the pole of an alternate-current electromagnet be held over the pole at a small distance from it, the electromagnetic repulsion is changed to an *attraction*. A description of this phenomenon, along with the suggested explanation, will be found in the *Electrician*, Vol. XXX., p. 603.

CHAPTER XI.

79. One of the most striking and important points of difference between continuous and alternating currents lies in the fact that whereas the former will only flow in a circuit which is wholly conducting, the latter will, on the other hand, flow in a circuit part of which is composed entirely of non-conducting material. The ability of an alternating current to traverse insulators gives rise to some interesting and somewhat complicated effects; many of these are usefully applied in practice. In considering effects of this nature, we have to deal with a class of phenomena which are entirely absent in the case of continuous currents.

80. The student who has been mainly concerned with the study of continuous-current phenomena will at this stage have to endeavour and enlarge the somewhat limited and one-sided view of matters electrical which such a study is apt to engender. He must learn to think of alternating currents as passing not only through conductors, but through such bodies as bits of glass or guttapercha, oils, air spaces, etc.—*i.e.*, through insulators or *dielectrics*—without destroying them or altering their nature; and to remember that in all such cases the current produces precisely the same *magnetic effect* as that which is characteristic of it when flowing in a conductor.

The mechanism by which a current is conveyed along a conductor—whatever the ultimate nature of that mechanism may be—is different from that by which it passes through a dielectric. In order to distinguish between the two modes of conduction, it is usual to refer to currents passing along conductors as *conduction currents*; the varying currents which are capable of traversing a dielectric being termed *displacement currents*. These terms are convenient, but the student must be warned against the notion that there are *two kinds* of current which differ from each other in their effects; a conduction current is in every way similar to, and produces the same effects as, a displacement current, and the two terms merely relate to the nature of the medium through which the (same) current passes.

Probably the best method of introducing the student to the study of displacement currents is to have recourse to a very rough but useful hydraulic analogy. This is accordingly the method which we shall follow in dealing with the subject.

81. In Fig. 59 let P A B represent a circular closed pipe, inside which is a piston P which fits it exactly, and which is capable of sliding freely inside the pipe. We shall suppose the pipe to be completely filled with a viscous incompressible liquid. If the piston, P, be caused to move round the pipe, a flow of the liquid will take place in the pipe, and this flow will be due to the difference of pressure on the two sides of the piston. If we suppose the liquid to be initially at rest, and a steady force to be applied to the piston, then, owing to the inertia due to the mass of the liquid, the

velocity will increase gradually until a steady flow is produced. While the flow is variable the force driving the piston does work in two ways: (1) in imparting kinetic energy to the moving mass of liquid; (2) in overcoming the resistance to motion which is due to the viscosity of the liquid. The first portion of the energy is stored up; the second is dissipated in producing heat. When the flow has become steady no further storage of energy takes place; the whole of the energy supplied is now being dissipated in the form of heat. The rate of flow (which is measured by the quantity of liquid that flows past any cross-section of the tube in unit of time) which results for a given driving force applied to the piston will depend on the resistance offered to the motion by the viscosity of the liquid; a less viscous liquid will attain a greater velocity than a more viscous one. Let us suppose that after the flow has been going on for some time the steady force propelling the piston is suddenly removed. Owing to the inertia of the liquid, the flow will not

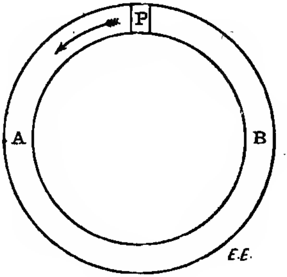

FIG. 59.

cease suddenly but will go on for a time, the liquid pushing the piston in front of it. The force propelling the piston is now derived, not from any external source, but from the kinetic energy stored up in the moving liquid. Since no further supply of energy takes place, it is evident that the motion will gradually subside, the kinetic energy of the liquid taking the form of heat.

82. For the pipe enclosing the liquid substitute a closed conducting circuit, for the piston a source of E.M.F., and we get a rough and very imperfect, but nevertheless useful, method of picturing to our minds the actions which are involved in starting, maintaining, and stopping a current in a circuit. The rate of flow of the liquid corresponds to the current, the viscosity of the liquid to the resistance of the circuit, the mass of the liquid to the inductance of the circuit. The energy stored up in the moving liquid corresponds to that stored up in the magnetic field produced by the current. In each case, while the flow lasts, we have the transformation of energy into heat.

83. As we shall pursue this analogy somewhat further, and apply it to the case of a circuit a portion of which consists of non-conducting material, it will be necessary to warn the student against the notion that the analogy is intended to represent the actual mechanism of conduction. Analogies are very useful if not abused and mistaken for the actual systems which they are intended to illustrate; but if pressed too far, they may become sources of danger, and give rise to utterly erroneous ideas. In order to show the student how imperfect our analogy is, and thus put

him on his guard, we shall draw his attention to one important matter in which our hydraulic analogy signally fails. The kinetic energy of the moving liquid has its seat in the liquid itself, so that the energy is contained within the pipe. But in the case of an electric circuit, we know that in most instances the magnetic field spreads out into the surrounding space, and the bulk of the stored energy will have its seat, not in the conductor itself, but in the space surrounding it (or, more correctly, in the *ether* which fills that space).*

84. Instead of the simple pipe shown in Fig. 59, let us next suppose that we have a similar pipe, but furnished with an elastic membrane, M (Fig. 60), which does not allow any flow of liquid to take place across it. Such a membrane would correspond to an insulator interposed between the two ends of a conducting circuit. If we suppose that the membrane is strong enough to resist any pressures which may be brought to bear upon it, then it is obvious that no continuous flow can take place in such a circuit. But since we have supposed the membrane to be elastic, it follows that if a force be applied to the piston, P, the pressure will be transmitted through the liquid in the

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* Another important point on which the hydraulic analogy is misleading is the following According to this analogy, the phenomenon of an electric current consists of actions which go on entirely inside the conductor; whereas according to modern views the most important actions which constitute an electric current take place *outside* the conducting circuit. The current is supposed to consist in the motion of tubes of electrostatic induction which, starting from the source of E.M.F., sweep across the space bounded by the circuit, the ends of the tubes gliding along the surface of the conductor, until the tubes enter the conductor, and thus become dissolved or destroyed.

direction of the arrow, and this pressure acting on the elastic membrane, M, will cause the latter to bulge (as shown by the dotted line), so that a *temporary* flow of the liquid will take place. Imagine the pressure to be gradually increased. The membrane will bulge more and more, and the *state of strain* produced in it will increase, until when a certain pressure is reached, the membrane gives way, being ruptured; when rupture of the membrane has taken place, a steady flow becomes possible.

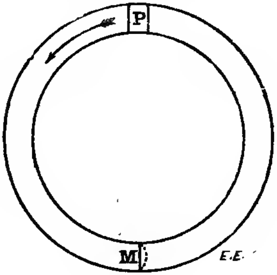

FIG. 60.

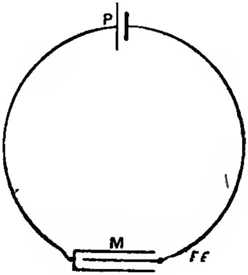

FIG. 61.

Returning now to the electrical case, let us suppose that we have a source of E.M.F., P (Fig. 61), connected to an arrangement, M, of two conductors separated by a dielectric; such an arrangement is generally called a *condenser*. When the E.M.F. is first introduced into the circuit there is some sort of elastic yielding of the dielectric which corresponds to the elastic yielding of the membrane in the hydraulic analogy. This yielding of the dielectric constitutes a *displacement current*. During the yielding a certain *quantity* of electricity (so called) passes into the condenser; this corresponds to the quantity of liquid which fills the space rendered vacant by the bulging of the membrane, M (Fig. 60). If the E.M.F. is gradually increased, the strain in the dielectric will increase in proportion, until ultimately the material breaks down, a spark passing through it; thus an *arc* is established between the two conductors, through which a continuous current now becomes possible.

85. The quantity of electricity which passes into a condenser when a P.D. is established between its terminals depends on the amount of elastic yield in the dielectric, and this latter quantity is in direct proportion to the applied P.D. Hence we may write

$$\text{quantity} = C \times \text{P.D.},$$

where C is a certain constant quantity independent of the P.D. This constant quantity is termed the *capacity* of the condenser. If Q, C, and V stand for the quantity, capacity, and P.D. of a condenser respectively, then the relation connecting these quantities is

$$Q = C \times V.$$

Since, when $Q=1$ and $V=1$, C is also equal to 1, it follows that a condenser of unit capacity is one which is charged to unit P.D. by unit quantity.

The practical unit of capacity is the *farad*, and it is defined as the capacity of a condenser which is charged to a P.D. of 1 volt by a quantity of 1 coulomb. For practical purposes the farad is too large a unit, and it is usual to express capacities in terms of the *microfarad*, which is one-millionth of a farad.

It is not difficult to deduce an expression for the amount of energy stored up in a condenser. The relation between the quantity and P.D. is represented by a straight line, such as O B in Fig. 62. Let us suppose that the condenser is charged to a P.D. of V C.G.S. units by a quantity of Q units (C.G.S.). If O A in Fig. 62 is made equal to Q, and A B to V, then C, the capacity of the condenser, is equal to $\frac{Q}{V} = \frac{OA}{AB}$. Let us suppose that at a certain stage during the charging the quantity is O T, the corresponding P.D. being T U; and let us calculate the work done in increasing the charge by a small amount T L. While a quantity T L is being introduced into the condenser, the P.D. rises from T U to L N, and its mean value between those limits is R S (T R being equal to R L). Now it follows from the definition of potential that in transferring T L units of quantity against a P.D. of R S units an amount of work is done which is represented by T L × R S. But the latter product represents the area of the strip T U N L. Hence, the work in ergs

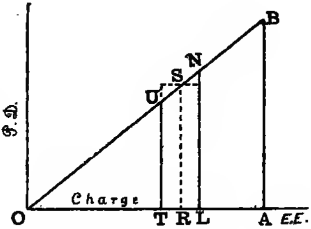

FIG. 62.

done in increasing the charge by an amount T L is represented by the area of the strip T U N L. Applying this result to each successive increment of the charge, we see that the total amount of energy stored up in the condenser is represented by the area of the triangle O A B, and hence is equal to $\frac{1}{2}$ O A × A B—*i.e.*, to $\frac{1}{2}$ Q V. This result may also be written in either of the forms $\frac{1}{2}CV^2$ or $\frac{1}{2}\frac{Q^2}{C}$.

86. Returning to Fig. 60, let us suppose in the first place that the liquid used is an extremely viscous one, offering a considerable resistance to motion. On applying a force to the piston, P, the liquid is caused to flow slowly, distorting the membrane, M, until the latter takes up a certain permanent position, depending on the difference of pressure on its two sides. Translated into the electrical case, this means that when a condenser is charged through a very high resistance the charging current is unidirectional. Suppose, in the second place, that we have a liquid of very high density—*i.e.*, possessing considerable *mass* per unit volume—and which is extremely mobile. If a constant force is suddenly applied to the piston, P, the liquid will be gradually set in motion, a (positive) flow taking place while the membrane is being distorted. But owing to the considerable inertia and absence of internal friction, the liquid will *tend to move on* after the membrane has taken up its equilibrium position. Hence the membrane will overshoot its mark, and a series of to-and-fro oscillations will take place, the flow of the liquid taking place in alternate directions and with gradually diminishing amplitude,

until the whole of the kinetic energy has become dissipated, and the membrane taken up its position of permanent distortion. The less the viscosity of the liquid, the longer will be the time throughout which these oscillations persist. It is also evident that while the oscillations are going on the difference of pressure on the two sides of the membrane alternates between values which are successively above and below that reached when a permanent distortion of the membrane has set it. Translating once more into the electrical case, we find that if a condenser is charged through a circuit of high inductance and very low resistance the charging current is not unidirectional, but *alternating*, with gradually diminishing amplitude; and the P.D. at the condenser terminals undergoes oscillations, rising above and falling below the E.M.F. of the charging battery.

87. Although the occurrence of certain actions in the hydraulic analogy is *no proof* that similar effects must occur in the electrical case, yet we may reason from the better known to the less known, and then leave it for direct experiment to decide whether the analogy has been pushed beyond legitimate limits or not. In the case under consideration experiment has shown that our deductions are correct for the electrical case; that under certain conditions oscillations such as we have been considering actually do take place.

When the time-current curve for an oscillatory charge is determined it is found to consist of positive and negative half-waves of gradually diminishing amplitude (but the same period), so that the area of each positive half-wave exceeds that of the succeeding negative half-wave, and therefore a certain quantity of electricity enters the condenser after each complete wave. The excess of the areas of the positive over those of the negative half-waves measures the charge (or quantity of electricity) in the condenser.

Since the P.D. at the condenser terminals during an oscillatory charge rises periodically to values in excess of the charging E.M.F., it follows that a condenser which is strong enough to stand a unidirectional charge from a given source of E.M.F. might break down with the same E.M.F. if the conditions of the circuit are such as to produce oscillations.

What has been said with regard to the *charging* of a condenser applies also to its *discharge*, and under certain conditions the discharge of a cylinder becomes oscillatory.

From the explanations given above it follows that although, so long as the elastic membrane remains intact, no *steady flow* can take place in a system such as that shown in Fig. 60, yet if the pressure on the two sides of the piston in Fig. 60, and the E.M.F. in Fig. 61, be made to alternate, the elastic membrane in the one case and condenser in the other *will respond* to the *alternating* pressure, and we thus see that an alternating (displacement) current is capable of passing through a dielectric.

88. In our hydraulic analogy, the rate of flow at any instant is measured by the quantity of liquid passing per second through any cross-section of the circuit. In the electrical case, the current at any instant is measured by the rate at which "quantity of electricity" is being transferred through any cross-section

of the circuit. But the quantity which is so transferred passes into the condenser. Hence *the current is measured by the rate of change* (*Art.* 5) *of the quantity in the condenser.* This result we shall frequently have to make use of in what follows.

Notes on Chapter XI.

The student will be helped towards a clearer understanding of the principles explained in the present chapter by referring to Chapters II. and III. of Dr. Lodge's "Modern Views of Electricity," where a number of mechanical models illustrating the actions taking place in a dielectric during charge and discharge are described. He is also recommended to read § 1 of Chapter V. of Dr. Fleming's "Alternate-Current Transformer."

Exercises.

1. A condenser of 10 microfarads capacity is charged to a P.D. of 300 volts. How many ft.-lbs. of energy have been stored up in the condenser?

2. A condenser whose capacity is 20 microfarads receives a charge of ·001 coulomb. It is then made to share its charge with another condenser, whose capacity is 10 microfarads. Find the energy, in ergs, stored up in the two condensers.

3. A battery whose E.M.F. is 100 volts is connected through a resistance to the terminals of a condenser for ·01 second, and is then disconnected. It is found that the P.D. across the condenser terminals is 80 volts. If the capacity of the condenser is 8 microfarads, what is the average value of the current which flowed into the condenser?

CHAPTER XII.

89. We are now in a position to study the behaviour of a condenser when placed on an alternating-current circuit, and shall first of all consider the simple case in which the circuit is made up of a condenser whose terminals are connected to a source of alternating E.M.F., the resistance and inductance of this source being negligible. We shall also assume that the impressed E.M.F. varies according to the simple harmonic law. From the first assumption it follows that the P.D. at the terminals of the condenser is at every instant equal to the impressed E.M.F.

Let e stand for the instantaneous value of the E.M.F., q for the instantaneous quantity, and C for the capacity of the condenser. Then we have the relation (Art. 85)

$$q = C\, e,$$

and since we suppose e to vary according to the simple harmonic law, it follows that q will vary according to the same law, and be in phase with e.

Let E stand for the maximum value of e, and Q for the maximum value of q. Then $Q = CE$. The E.M.F. may be represented by an alternating vector (Art. 15) derived from the rotating vector O A (Fig. 63), of length E. The quantity of electricity in the condenser may be represented by another alternating vector, which is in phase with the first, but whose maximum value is C E. Next, since the current is measured by the rate of change of the quantity (Art. 88), and since the latter is represented by an alternating vector, it follows that the current itself will also be represented by an alternating vector, of amplitude p C E (Art. 22), this vector being $\frac{\pi}{2}$ in advance of the quantity vector as regards phase. The letter p stands, as usual, for 2π times the periodicity.

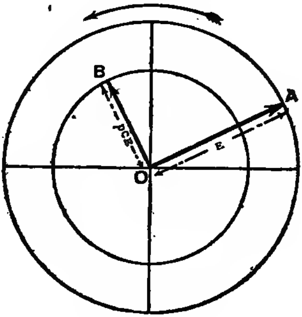

FIG. 63.

If, therefore, in Fig. 63 we draw O B $= p$ C E at right angles to O A, and if we suppose the whole diagram to revolve round O with constant angular velocity p, then the projections of the rotating vectors O A and O B on the vertical axis will give us the instantaneous values of the E.M.F. and current respectively. We thus arrive at the following important result:

If a simple harmonic P.D. of amplitude E be applied *to a condenser of capacity C, then the current through the condenser will be of amplitude p C E, and will be* $\frac{\pi}{2}$*—i.e., a quarter-period—in advance of the P.D. as regards phase.*

90. It may be well, perhaps, at this point to anticipate a paradox which might possibly occur to the student and cause some difficulty. According to the result just established the current has its maximum value when the E.M.F. passes through the zero value. Let us suppose that there is a switch in the circuit and that the switch is closed at the instant of zero value of the E.M.F. Now, since the E.M.F. is just passing through its zero value it does not matter whether, at the instant considered, the switch is closed or not. From this it would appear to follow that there is a current through the condenser before the closing of the switch. What really happens is this. When the switch is closed the current starts from a zero value, but the first few waves are of an abnormal type; the current gradually gains on the E.M.F. as regards phase, and ultimately reaches a phase difference of $\frac{\pi}{2}$. Our investigations are concerned with the steady periodic state, and not with the abnormal waves which make their appearance on first closing the circuit.

A similar difficulty occurs in the case of an inductive circuit, and is similarly met.

91. The result established for a simple harmonic E.M.F. wave may, by the principles of Arts. 43-45, be extended to waves of any shape whatever. The method is the same as that used in working out the

numerical example of Art. 46. We shall take the same E.M.F. as that used in Art. 46—viz.:

$$e = 2{,}000 \sin p\,t - 600 \sin 3p\,t,$$

and, assuming a periodicity of 100 ∼ per second, suppose that this E.M.F. is applied to a condenser of capacity = 15·9 microfarads. We then have

$$p = 2\,\pi\,n = 200\,\pi\,; \;\; 3p = 600\,\pi.$$

$$p\,\mathrm{C} = 200\,\pi \times 15{\cdot}9 \times 10^{-6} = {\cdot}01 \text{ approximately},$$

and $$3\,p\,\mathrm{C} = {\cdot}03.$$

Hence, taking the first term in the expression for the E.M.F.—*i.e.*, 2,000 sin $p\,t$—we find that the corresponding current is represented by $20 \sin\left(p\,t + \frac{\pi}{2}\right)$; while the current corresponding to the second term, $- 600 \sin 3p\,t$, is given by $- 18 \sin\left(3\,p\,t + \frac{\pi}{2}\right)$. The total current through the condenser is therefore represented by

$$20 \sin\left(p\,t + \frac{\pi}{2}\right) - 18 \sin\left(3\,p\,t + \frac{\pi}{2}\right).$$

In Fig. 64 the chain-dotted curve is the E.M.F. curve (the harmonic components of which are shown in Fig. 37). The full-line curve is the current curve, whose harmonic components are shown dotted. The student will at once observe that the current curve departs much more widely from the simple sine form than does the E.M.F. curve; so that the effect of a condenser on the current curve is to produce still greater distortion than that which is already present in the E.M.F. curve. In this respect the effect of a condenser is the reverse of that produced by an inductance, as the student will readily see by referring to Art. 46. An inductance may be said to have a smoothing effect on the current wave, and a capacity a distorting effect. The reason for this is not far to seek. In the case of an inductance, the current, for a harmonic E.M.F. of given amplitude, varies inversely as the periodicity $\left(I = \frac{E}{Lp}\right)$; but in the case of a capacity, it varies directly as the periodicity ($I = p\,C\,E$). Hence an inductance will tend to

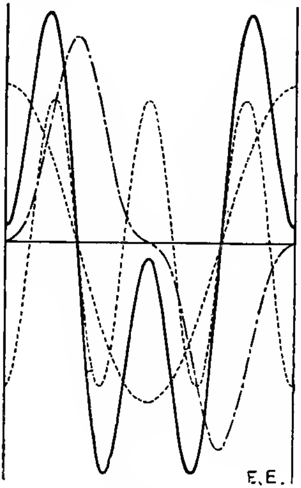

FIG. 64.

suppress the higher harmonics in the current wave, whereas a capacity will tend to bring them into greater prominence.

92. Returning to the case of a simple harmonic E.M.F., we see that, by the result established in Art. 88, the relation between the P.D. at the terminals of a condenser and the current through it is given by the equation

$$I = p\,C\,E \qquad . \quad . \quad . \quad . \quad . \quad . \qquad (25)$$

where I and E may each stand for either the maxima or the R.M.S. values of the current and the P.D. respectively. It will be instructive to compare and contrast the behaviour of a simple inductance with that of a simple capacity when the impressed E.M.F. follows the simple harmonic law. This is done in the following table:

Simple Inductance.	*Simple Capacity.*
The current is $\frac{\pi}{2}$ *behind* the E.M.F. as regards phase.	The current is $\frac{\pi}{2}$ in *advance* of the E.M.F. as regards phase.
The current varies *inversely* as the product of the inductance and periodicity.	The current varies *directly* as the product of the capacity and periodicity.

Since the effects produced by an inductance are thus seen to be the very reverse of those produced by a capacity, the idea at once suggests itself that a capacity may be used to neutralise the effects of an inductance. As we shall see presently, this may be accomplished when the E.M.F. obeys the simple harmonic law.

93. Let us suppose that a circuit consists of an inductive resistance connected in series with a condenser, as shown in Fig. 65. Let r and L be the resistance and inductance of the circuit respectively and C the capacity of the condenser. Let us suppose that the current obeys the simple harmonic law, and that its amplitude is I. The problem is to find the impressed E.M.F. We may regard the latter as made up of three components: (1) the $r\,i$ component; (2) the $L\,i'$ component (Art. 37); (3) the P.D. at the condenser terminals.

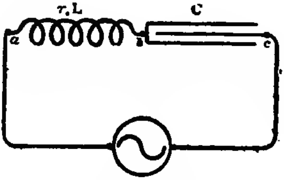

FIG. 65.

In Fig. 66, O L is a rotating vector of length I. The alternating vector derived from this by projecting it on the vertical axis gives the current. O M is a rotating vector of length $r\,I$, and coincident in direction with O L; the projection of this vector at any instant on the vertical axis gives the $r\,i$ component of the E.M.F. at that instant. The $L\,i'$ component is represented by the projection on the vertical axis of a rotating vector O P, of length $L\,p\,I$ drawn at right angles to O L (Art. 37). Lastly, the condenser P.D., which by the results established in Art. 89 is $\frac{\pi}{2}$ behind the current as regards phase, is represented by the projection of a rotating vector O R, of length I/C p, drawn at right angles to O L, but in a direction opposite to that of O P.

The whole diagram is, of course, supposed to revolve in a counter-clockwise direction (as shown by the arrow) round O, the angular velocity being p (where $p = 2\pi n$).

The instantaneous impressed E.M.F. is the sum of the projections of the rotating vectors O M, O P, and

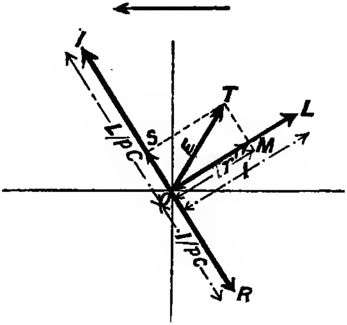

Fig. 66.

O R on the vertical axis. But this sum is (Art. 12) the same as the projection of the resultant of the three rotating vectors. In order to find this resultant, we may first of all find the resultant of O P and O R. If we assume that O P > O R, then the resultant of these two vectors will be represented by a vector O S, of length L p I − I/C p, and having the same direction as O P. By compounding the vectors O S and O M

(Art. 11), we find the required resultant O T of the three vectors.

We thus see that the impressed E.M.F. is represented by the projection on the vertical axis of the rotating vector O T, and from the geometry of the figure we can at once determine the maximum value of the impressed E.M.F. and the angle of phase difference T O L between it and the current. Noticing that $OM = r\,I$, and $MT = OS = \left(Lp - \frac{1}{Cp}\right) I$, we get

$$OT = \sqrt{OM^2 + MT^2}$$

$$I = \sqrt{r^2 + \left(Lp - \frac{1}{Cp}\right)^2}$$

Hence, denoting by E the maximum value of the impressed E.M.F., we see that

$$I = \frac{E}{\sqrt{r^2 + \left(Lp - \frac{1}{Cp}\right)^2}} \quad . \quad . \quad (26)$$

Again, if θ stand for the angle of lag T O L of the current behind the impressed E.M.F., we have

$$\tan\theta = \frac{MT}{OM} = \frac{Lp - \frac{1}{Cp}}{r} \quad . \quad . \quad . \quad (27)$$

94. Several important results follow at once from the relations just established. If $Lp = \frac{1}{Cp}$, then we get $I = \frac{E}{r}$, and $\theta = 0$. In other words, by choosing

the capacity so that it is equal to $\frac{1}{\mathrm{L}p^2}$, we cause the current to come into phase with the impressed E.M.F., and to assume a value which is simply equal to the quotient of the E.M.F. by the resistance of the circuit. The effect of inductance is thus completely annulled by capacity.

Again, a glance at Fig. 66 shows that the current may lag, be in phase with the E.M.F., or lead, according as $\mathrm{L}\,p$ is greater than, equal to, or less than $\frac{1}{\mathrm{C}p}$.

Further, we have the important result that under certain conditions *the P.D. at the condenser terminals may considerably exceed the impressed E.M.F.* in the circuit, so that a voltmeter connected across the points *b* and *c* (Fig. 65) will give a higher reading than one connected across *a* and *c*. This is evident from the fact that for certain values of the resistance, inductance, and capacity the vector O R may exceed O T.

Similarly, *it may happen that the P.D. across the inductive resistance exceeds the impressed E.M.F.* For the vector O P may under certain conditions exceed O T. If this is the case, a voltmeter connected across *a b* will give a higher reading than one connected across *a c* (Fig. 65).

95. If the periodicity is so related to the inductance and capacity that the condition $\mathrm{L}\,p = \frac{1}{\mathrm{C}p}$ is fulfilled, then it follows, by the last article, that the current will be in phase with the impressed E.M.F. This pheno-

menon of the neutralisation of an inductance by means of a capacity is generally referred to as *electrical resonance.* As an example, let us suppose that it is required to neutralise an inductance of ·0127 henry, when the periodicity is 100 ∿ per second. Let C stand for the capacity required. Then we have

$$\mathrm{L}p = \cdot 0127 \times 2\pi \times 100 = 79 \cdot 5.$$

Hence we must have $\mathrm{C}p = \dfrac{1}{\mathrm{L}p} = \cdot 01258,$

so that $\mathrm{C} = \dfrac{\cdot 01258}{200\pi} = \dfrac{\cdot 01258}{628 \cdot 32} = 2 \times 10^{-5}$ farads approximately, or C = 20 microfarads.

Let us suppose, further, that the resistance of the circuit is 10 ohms, and that the impressed E.M.F. is 100 volts. The current is in this case $\dfrac{100}{10} = 10$ amperes. The P.D. at the condenser terminals is $\dfrac{10}{\mathrm{C}p} = 79 \cdot 5 \times 10 = 795$ volts, or almost *eight times as great as the impressed E.M.F.* The P.D. across the inductive resistance is $10\ \sqrt{r^2 + p^2 \mathrm{L}^2} = 10 \times 80 \cdot 1 = 801$ volts.

We therefore see that a comparatively small impressed E.M.F. may under suitable conditions give rise to very high P.D.'s between certain points in the circuit. It is important to bear this fact in mind, for neglect of the necessary precautions might in some cases result in the breaking down of costly apparatus which was never intended to stand the P.D. to which it was unwittingly subjected.

Excessive rises of P.D. in a circuit may be prevented by the introduction of sufficiently large non-inductive resistances. Thus, if in the example just considered we suppose that the resistance in the circuit is 50 ohms instead of 10, the current will be 2 amperes, and the P.D. at the condenser terminals only $79 \cdot 5 \times 2 = 159$ volts instead of 795 as in the first case.

If a rise of P.D. is not objectionable, then we may take advantage of the phenomenon of electrical resonance to increase the current in an inductive circuit when the available impressed E.M.F. is insufficient for the purpose.

96. We shall next consider the effect of shunting an inductive resistance by a condenser. If the student has followed the explanation given above for the case of a condenser placed in series with an inductive resistance, he will have no difficulty in understanding what follows.

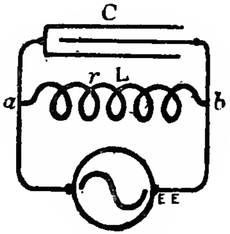

FIG. 67.

Let r and L denote the resistance and inductance of the circuit, and C the capacity of the shunting condenser. The arrangement of connections is shown in Fig. 67. Let E stand for the amplitude of the simple harmonic P.D. which is maintained between the points

a and b. The problem is to find the currents in the various branches of the circuit.

The amplitude I_1 of the current in the inductive part of the circuit is, by Art. 38, equation (20), equal to $\frac{E}{\sqrt{r^2 + p^2 L^2}}$. Proceeding as in Art. 93, we lay off a vector O L (Fig. 68), of length I_1; a vector O M, of

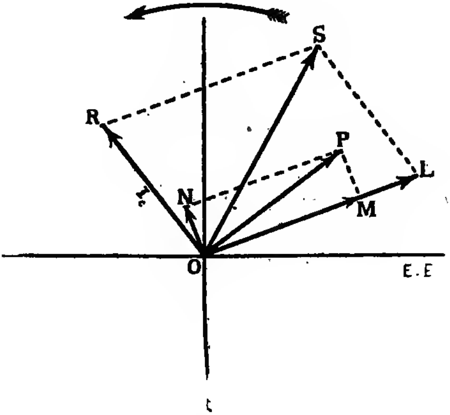

Fig. 68.

length $r\,I_1$, and at right angles to this a vector O N, of length $L\,p\,I_1$. The resultant of O M and O N is O P, whose length is E. Let this diagram of vectors revolve round O with constant angular velocity p. Then the projections of the vectors O L, O M, O N, and O P on the vertical axis at any instant give the instantaneous values of the current in the inductive part of the circuit, the $r\,i$ component of the impressed P.D., the $L\,i'$ component, and the total impressed P.D.

respectively. Since the capacity of the shunting condenser is C, the amplitude of the condenser current I_c is, by Art. 92, equation (25), equal to p C E, and we know that this current is $\frac{\pi}{2}$ in advance of the impressed P.D. as regards phase. Hence, if we draw a vector O R, of length I_c, at right angles to O P, the projection of this vector on the vertical axis at any instant will give the value of the condenser current. Now, the total current which comes from the source of alternating E.M.F. is at every instant equal to the algebraical sum of the currents through the inductive part of the circuit and through the condenser. Hence the total current at any instant is equal to the sum of the projections of O L and O R on the vertical axis, or, which comes to the same thing, the projection of O S on the vertical axis; O S being the resultant of O L and O R.

97. From the diagram, Fig. 68, it is evident that the current in the main circuit may be either greater or less than the current through the inductive resistance. Let us suppose that the resistance and inductance of the circuit are fixed, but that we have the means of varying the capacity of the shunting condenser. The amplitude of the impressed P.D. is also supposed to be fixed. This is equivalent to saying that the vectors O L, O M, O N, and O P are all given, but that the length of the vector O R may be varied (the direction of O R is fixed, since it must always be at right angles to O P). The length of the vector O R will evidently be in direct proportion to the capacity of the shunting condenser, since $I_c = p\,C\,E$. When the capacity is

nothing—*i.e.*, when the condenser is entirely removed—the vector O S will be coincident with O L, which is another way of expressing the fact that the current in the main circuit is the same as that in the inductive resistance. As the capacity is gradually increased, the extremity S of the vector O S will travel along the straight line L S, which is parallel to O R. The current* in the main circuit will therefore gradually diminish and *reach a minimum value* when O S becomes perpendicular to L S (or coincident *in direction* with O P)—*i.e.*, when the *current in the main circuit comes into phase with the impressed E.M.F.* If the capacity is still further increased, the vector O S will increase in length and at a certain stage become equal to O L; it will then go on increasing indefinitely as the capacity is increased. Now, it is evident from the diagram that for all values of the current in the main circuit which lie between the *minimum* value of that current and its value when the shunting capacity is zero, we may have *two* different values of the capacity. With the *smaller* value of the capacity, the current will *lag*, the vector O S lying between O L and O P; with the *larger* value, the current will *lead* by the same amount, the vector O S lying on the other side of O P.

We shall work out a numerical example to illustrate this point. Let $r = 6$ ohms, $L = \cdot 01273$ henry, $E = 100$ volts, and let the periodicity be 100 ∼ per sec. The impedance is $\sqrt{r^2 + p^2 L^2} = 10$, hence $I_1 = \frac{100}{10} = 10$. If θ denote the angle of lag between the impressed

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* *i.e.*, its R.M.S. value.

P.D. and the current in the inductive resistance, then, by Art. 38, equation (21), $\tan\theta = \frac{Lp}{r} = \frac{8}{6} = \frac{4}{3}$. Take a line O L (Fig. 69), of length 10, and at L draw L A at right angles to O L, making L A $= \frac{4}{3}$ O L. Then $\tan \text{L O A} = \frac{\text{L A}}{\text{O L}} = \frac{4}{3} = \tan\theta$. Hence the angle L O A is equal to the angle of lag θ. From L draw

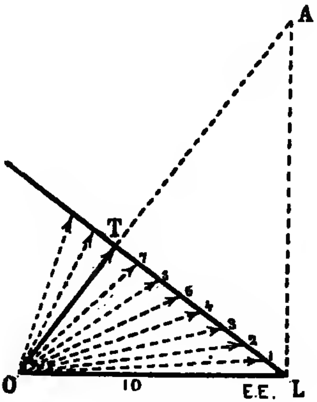

FIG. 69.

L T at right angles to O A. The lines O L and L T in Fig. 69 correspond to O L and L S in Fig. 68. When there is no shunting condenser, the maximum value of the current in the main circuit is the same as in the inductive resistance—viz., 10—and the angle of lag is L O A.

Let us now introduce a shunting condenser and gradually increase its capacity. Suppose we consider values of the capacity corresponding to 1, 2, 3, etc., amperes of maximum condenser current. Since $I_c = pCE$, we get $C = \frac{I_c}{pE} = \frac{I_c}{200\pi \times 100}$; corresponding, therefore, to 1 ampere of the maximum condenser current we have a capacity of 15·92 microfarads, corresponding to 2 amperes, 2 × 15·92 or 31·84 microfarads, etc. In order to determine the effect of this condenser current on the current in the main circuit, we lay off lengths L 1, L 2, L 3, etc., to represent 1, 2, 3, etc., amperes respectively, and join the point O to the points 1, 2, 3, etc. The lines O 1, O 2, O 3, etc., give the values of the maximum current I in the main circuit, and the angles 1 O T, 2 O T, 3 O T, etc., the corresponding values of the angle of lag between the impressed E.M.F. and the current in the main circuit. Taking corresponding values of the capacity and the amplitude of the current in the main circuit, we may

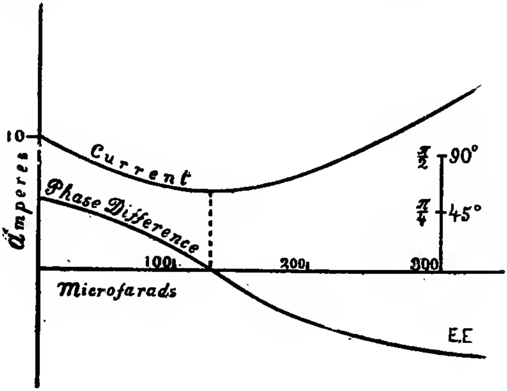

FIG. 70.

plot a curve as in Fig. 70, showing the relation between these two quantities. Another curve, shown in the same figure, may be plotted to exhibit the connection between the capacity of the shunting condenser and the phase difference between the impressed E.M.F. and the current in the main circuit. The angle of phase difference is reckoned *positive* if the current *lags*, *negative* if it *leads*.

98. The action of a condenser when connected as a shunt to an inductive resistance may be thus explained in general terms. The total current in the inductive circuit may be analysed into two components: the *load* current, which is in phase with the impressed E.M.F., and the *idle* current, which is in quadrature with it (Art. 62). Now, this idle current is $\frac{\pi}{2}$ behind the E.M.F. as regards phase, and the current of the shunting condenser is $\frac{\pi}{2}$ in advance of it. Hence these two currents are always of opposite sign, so that when the idle current is, say, flowing from *a* to *b* (Fig. 67), the condenser current is flowing from *b* to *a*. We thus have a *local current in the branched circuit which does not pass into the main circuit at all.* If the capacity of the condenser is so adjusted that its current is equal to the idle current of the inductive resistance, then it is obvious that the only current in the main circuit will be the load current. *The load current represents the minimum value which the actual current in the main circuit may have.* For if the idle current in the inductive resistance either exceeds or falls below the condenser current, then the excess of current must

pass into the main circuit, and hence the current in the latter will be increased.

There is a fallacy connected with this problem of a shunting condenser to which it may be well to allude. It is the notion that the condenser in some mysterious way becomes a source of power, and developes energy of its own accord. All the condenser does, however, is simply *to relieve the main circuit of the idle current.* The energy which is stored up in the magnetic field of the circuit, instead of surging between the inductive resistance and the source of E.M.F., surges between the former and the condenser. The condenser, in fact, acts as a convenient reservoir for the discharge of the stored energy from the inductive circuit, this energy being subsequently returned to the circuit.

99. Advantage is taken of this effect in some cases for the purpose of increasing the power factor of a circuit (Art. 59).* Thus, let us suppose that a 6,000-watt transformer of the open magnetic circuit type is found to take a primary current of 1·194 amperes when connected across 2,400-volt mains, the secondary circuit being open; and that the *true watts* (Art. 59) as measured by a wattmeter amount to 151. The *apparent watts* are in this case equal to 2,400 × 1·194 = 2,866 watts, so that the power factor is only $\frac{151}{2,866}$ = ·0527. The problem is to find the capacity of a shunting condenser which will render the power factor for the mains equal to unity, the periodicity being 83 ∾ per second.

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* The advantages resulting from working with a high power factor have already been briefly considered in Art. 60.

In working out this example we shall assume that the E.M.F. and currents obey the simple sine law—an assumption which would in most cases not be justified in practice. But in the case of problems like the one in hand such an assumption leads to results which are sufficiently correct for practical purposes.

The relation between the total current, the load current, and the idle current (Art. 62) may be represented by the three sides of a right-angled triangle. If, therefore, two of these quantities are known we can at once determine the third. Thus, referring to Fig. 71, we see that, the triangle being right-angled,

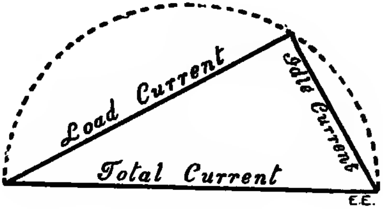

FIG. 71.

$$\textit{idle current} = \sqrt{(\textit{total current})^2 - (\textit{load current})^2}.$$

In our example the total current is 1·194; the load current is $\frac{151}{2{,}400} = {\cdot}0629$. Hence

$$\text{idle current} = \sqrt{1{,}194^2 - {\cdot}0629^2} = 1{\cdot}192 \text{ amperes.}$$

We have therefore to provide a condenser which will take this idle current of 1·192 amperes at 2,400 volts and a periodicity of 83 ∿ per second.

By Art. 92, equation (25), we have, if C denote the capacity of the condenser,

$$C = \frac{1{\cdot}192}{p \times 2{,}400}.$$

Now, $p = 2\pi \times 83$, and we find $C = {\cdot}9513 \times 10^{-6}$ farads, or ·9513 microfarad. A shunting condenser of this capacity will therefore render the power factor for the mains equal to unity, so that the current which flows into the transformer from the mains will only be ·0629 ampere, instead of 1·194 amperes, its value when the condenser is removed.

100. In all the above investigations we have assumed that the condenser current is a quarter-period $\left(\frac{\pi}{2}\right)$ in advance of the impressed P.D. at its terminals. If this were strictly true, then a condenser would absorb absolutely no power (Art. 61, Prop. II.) and the whole of the energy which is stored up in it during the first half-period, when the current is positive and the charge in the condenser is increasing, would be returned to the source of E.M.F. during the next half-period, when the current is negative and the charge diminishing. As a matter of fact, however, a condenser *does* absorb a certain amount of power, so that there is a small component of the current which coincides in phase with the impressed P.D., and hence the actual current is a little less than $\frac{\pi}{2}$ in advance of the P.D. A condenser, in fact, behaves as if it were a combination of a *perfect condenser—i.e.*, one which causes no dissipation of energy—with a small non-inductive resistance connected in series with it; this

equivalent resistance may be termed the *spurious resistance* of the condenser. The small load current of a condenser goes towards producing heat in the dielectric, which gets warmed up by the repeated alternating stresses to which it is subjected.

101. The effects which we have been considering may be briefly summed up by saying that a condenser of suitable capacity connected in series with an inductive resistance may be used for producing a rise of P.D. across the terminals of this resistance; and a condenser connected as a shunt to an inductive resistance may be used for supplying the idle current required by that resistance.

Cases more complicated than those we have dealt with may present themselves, but the student who has followed the preceding explanations will have no difficulty in obtaining, by means of suitably constructed vector diagrams, similar to those of Figs. 66 and 68, the solution of any problem involving combinations of a resistance, an inductance, and a capacity.

Notes on Chapter XII.

The most important practical uses of condensers are in telegraphy and telephony. In electric lighting and power distribution, they are not very extensively employed. At a time when transformers of the open magnetic circuit type were still manufactured, attempts were made to improve the power factor of the station during the hours of light load by connecting condensers of suitable capacity across the terminals of the transformers. At present, the uses of condensers in electric power plants may be said to be confined to: (*a*) the production of two alternating currents differing in phase from a single alternating current; (*b*) the improvement of the power factor in some types of alternate-current motors.

But although the use of condensers is, at any rate at present (owing, in no small degree, to their cost), somewhat restricted, we frequently have to consider the effects of capacity in alternate-current circuits; for there are many parts of a lighting or power plant which, although not intended to act as condensers, virtually do so. Thus, every concentric cable constitutes a condenser. The effects of capacity in alternate-current working probably first came to be generally appreciated after the description of the remarkable phenomena observed during the working of the Ferranti mains between Deptford and London. Since that time, somewhat more attention has been paid by engineers to questions of capacity.

An account of some interesting experiments bearing on the subject-matter of the present chapter will be found in a paper by Dr. Fleming, entitled "On Some Effects of Alternate-Current Flow in Circuits having Capacity and Self-induction," published in the *Journal of the Institution of Electrical Engineers*, Vol. XXI., pp. 372—376.

Exercises.

1. What must be the alternating P.D. between the terminals of a condenser of 1 microfarad capacity in order that a current of ·8 ampere may flow through the condenser, the periodicity being 120 ∼ per second?

2. A P.D. of 500 volts, alternating with a periodicity of 83 ∼ per second, is maintained between the terminals of a condenser whose capacity is 5 microfarads. Find the current.

3. A circuit is made up of an inductive resistance (5 ohms, ·02 henry) connected in series with a condenser of 20 microfarads capacity. An alternating E.M.F. of 300 volts, at a periodicity of 100 ∼ per second, is introduced into the circuit. Find the current and the P.D.'s across the inductive resistance and the condenser respectively.

4. The inductance of a circuit is ·35 henry, and it is found that an alternating E.M.F. of 100 volts produces a current of ·4 ampere, the periodicity being 100 ∼ per second. It is desired to increase the current to 1 ampere by connecting a condenser in series with the inductive resistance, the E.M.F. remaining unaltered. What must be the capacity of the condenser?

5. A certain alternating-current motor takes a current of 5 amperes at 500 volts, the periodicity being 80 ∼ per second. The power factor of the motor is ·8. What must be the capacity of a shunting condenser in order to increase the power factor to ·9?

CHAPTER XIII.

102. When two circuits are so placed relatively to each other that a current flowing in one of them produces a magnetic flux through the other, we say

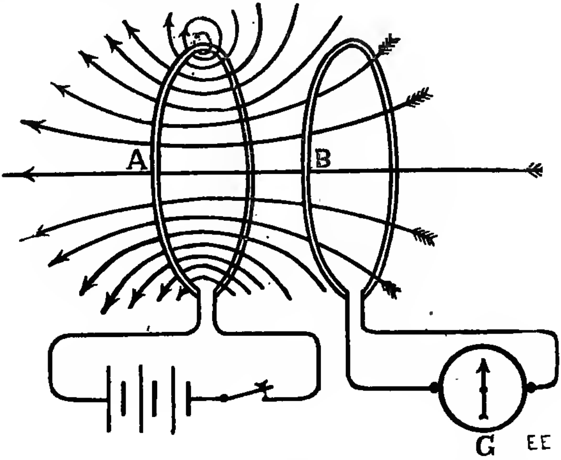

Fig. 72.

that there is *mutual inductance* between the circuits. For example, if we have two coils of wire, A and B (Fig. 72), one of which, A, may be connected to a battery, and the other, B, is permanently connected to a galvanometer, G; then by closing the circuit of A we poke a certain number of lines through B, thus giving rise to an induced E.M.F. in B and (since the circuit is closed) current through the galvanometer. The current is, of course, only momentary, and ceases as soon as the current in A has become steady (Art. 23). If the circuit is now broken, the lines due to the current in A are removed from B, and we get a momentary deflection of the galvanometer in the opposite direction.

By altering the position of B, we can increase or diminish the galvanometer deflection due to the making or breaking of the circuit A. In particular, a number of positions may be found for which there is *no deflection* when the circuit A is either made or broken. It will be found that in such positions the planes of the coils are mutually perpendicular, as shown in Fig. 73. For any such position, the magnetic lines proceeding from A run in a direction parallel to the plane of B, and therefore do not thread themselves through B. Hence the starting or stopping of a current in A does not give rise to any

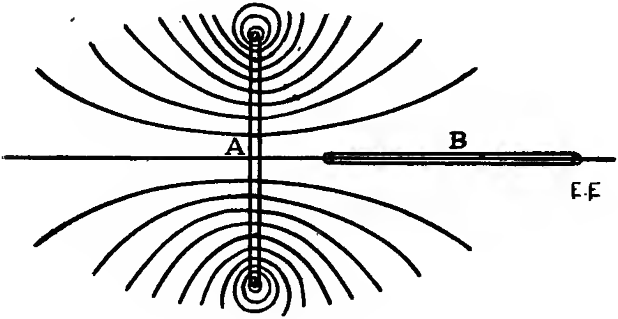

FIG. 73.

induced E.M.F. in B. When this is the case, we say that the two circuits are devoid of mutual inductance.

103. In order to investigate the effects produced by the mutual inductance of two circuits, it becomes necessary to adopt a unit in terms of which this quantity may be measured. Before doing so, however, we shall prove the following important proposition:

If two circuits, A and B, are so placed relatively to each other that a current i in A gives rise to N lines through B, then a current i in B will give rise to N lines through A.

Let us suppose that the number of lines through A due to a current i in B is N′. Let the current in each circuit be maintained constant and equal to i, and let the circuit B be gradually moved away from A, until it has travelled so far away from A that practically the whole of the lines through it due to the current in A have disappeared. In moving the circuit B away from A, a certain amount of work will obviously be done,* for during its motion the circuit B is cutting lines proceeding from A. The circuit B has a current i flowing in it, and in moving it away from A we have caused it to cut N magnetic lines. Hence by Art. 32 the work done in separating the circuits is N i.

Let us next suppose that instead of moving B we keep B fixed, and move A to a very great distance. The work done will in this case be $N' i$.

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* If we suppose that the fields due to the two currents assist each other. But if they oppose each other, then the E.M.F. induced in B during the motion will be a counter E.M.F., and will oppose the current, so that work will be done by B and not spent upon it as in the first case.

But the work done in moving B away from A must obviously be the same as that done in moving A away from B, since for any given distance between the two circuits the force exerted by A on B must equal that exerted by B on A. Hence we must have

$$N i = N' i,$$

or

$$N = N',$$

which establishes the proposition stated above.

104. Using this result, we define the *mutual inductance* (or *coefficient of mutual induction*) between two circuits as follows:

The mutual inductance of two circuits is the magnetic flux through one of them due to unit current in the other.

105. We are now in a position to investigate the establishment of a current under a steady impressed E.M.F. in a circuit which, besides having (self) inductance, has also mutual inductance.

Referring to Fig. 74, let us suppose that a steady E.M.F. of amount E is applied to the circuit A, of resistance r_1, and inductance L_1. Let the resistance and inductance of B be r_2 and L_2 respectively, and let the mutual inductance of the two circuits be M. The problem is to find the law of growth of the current in A.

Let i_1 and i_2 stand for the instantaneous values of the currents in the circuits A and B respectively, and let i_1' and i_2' stand for their rates of change (Art. 33). Consider, in the first place, the various E.M.F.'s which we have in the circuit A. These are: (*a*) the r_1 i_1

E.M.F., which is required to maintain the current i_1 through the resistance r_1; (*b*) the $L_1\ i_1'$ E.M.F., required to balance the counter E.M.F. due to the inductance of the circuit; (*c*) the $M\ i_2'$ E.M.F., which is required to balance the induced E.M.F. in A *due to the variation of the current in B*; (*d*) the impressed E.M.F., E. This latter must obviously be equal to the sum of the former three, so that

$$r_1 i_1 + L_1 i_1' + M i_2' = E \quad . \quad . \quad . \quad (a)$$

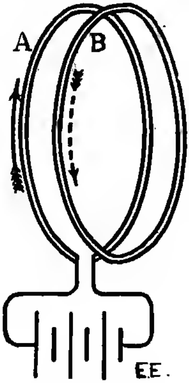

FIG. 74.

Similarly, in the circuit B we have the three E.M.F.'s $r_2 i_2$, $L_2 i_2'$, and $M i_1'$. Since there is no impressed E.M.F. in B, we have

$$r_2 i_2 + L_2 i_2' + M i_1' = 0 \quad . \quad . \quad . \quad (\beta)$$

If the values of i_1 and i_2 at any instant are known, then we can calculate the corresponding values of i_1' and i_2' from the above equations. For, multiplying equation (*a*) by L_2, and equation (β) by M and subtracting the latter from the former, we find

$$i_1' = \frac{L_2 E + M r_2 i_2 - L_2 r_1 i_1}{L_1 L_2 - M^2}.$$

Substituting this value for i_1' in equation (β), we get

$$i_2' = -\frac{M E - M r_1 i_1 + L_1 r_2 i_2}{L_1 L_2 - M^2}.$$

106. By using the above equations, and applying a step-by-step process similar to that used in Art. 36, we are able to draw approximately the curves showing the growth of the current i_1 in circuit A, and the growth and decay of the current i_2 in circuit B. We proceed to illustrate the method by a numerical example.

Let an E.M.F. of 100 volts be applied to a circuit of resistance 10 ohms and inductance ·05 henry, another circuit, of resistance equal to 5 ohms, and inductance 0·4 henry, being placed in the neighbourhood of the first circuit and in such a position that the mutual inductance is ·02 henry. Substituting these values in the expressions for i_1' and i_2' obtained above, we get

$$\left.\begin{aligned} i_1' &= 2{,}500 + 62{\cdot}5\, i_2 - 250\, i_1 \\ i_2' &= -(1{,}250 - 125\, i_1 + 156{\cdot}25\, i_2) \end{aligned}\right\} \quad . \quad (\gamma)$$

At the commencement, when the time $t = 0$, we have $i_1 = 0$, and $i_2 = 0$, so that the initial rates of current growth in the two circuits are $i_1' = 2{,}500$ and $i_2' = -1{,}250$. We shall assume that these rates of growth remain constant until i_1 reaches, say, $\frac{1}{20}$th of its steady value—*i.e.*, ·5 ampere. Since i_1 grows at the rate of 2,500 amperes per second, the time taken by i_1 to reach a value of ·5 ampere will be $\frac{\cdot 5}{2{,}500} = {\cdot}0002$ second. At the same time, the current i_2 has been growing at the rate of − 1,250 amperes per second, and, therefore, the value reached by it at the end of the ·0002 second is $-1{,}250 \times \cdot 0002 = -\cdot 25$ ampere. At the end of that interval, we take into account the fact that the rates of growth of the two currents do not remain constant, and, using the values $i_1 = \cdot 5$ and $i_2 = -\cdot 25$, we calculate from equations (γ) the new rates of growth. We thus get

$i_1' = 2{,}500 - 62{\cdot}5 \times \cdot 25 - 250 \times \cdot 5 = 2{,}359$ approximately, and $i_2' = -(1{,}250 - 125 + \cdot 5 - 156{\cdot}25 \times \cdot 25) = -1{,}148$ approximately.

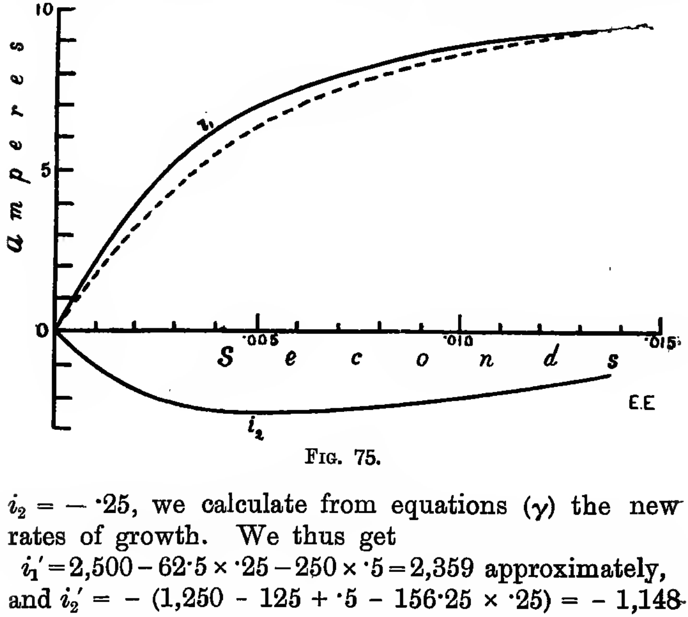

FIG. 75.

As before, we suppose that these new rates of growth remain constant until i_1 has again increased by ·5 ampere. This will take $\dfrac{\cdot 5}{2{,}359} = \cdot 000212$ second approximately. During the same time i_2 will have increased by $-1{,}148 \times \cdot 000212 = -\cdot 243$ ampere. Corresponding, therefore, to $t = \cdot 0002 + \cdot 000212 = \cdot 000412$ second, we have the values $i_1 = 1$ and $i_2 = -\cdot 493$. Substituting these new values of i_1 and i_2 in equations (γ) we once more calculate the corresponding values of i_1' and i_2', etc.

Proceeding in this way, we obtain the following set of values:

$t =$	0	·0002	·000412	·000637	·000878	·001135	·001413
$i_1 =$	0	·5	1·0	1·5	2 0	2·5	3·0
$i_2 =$	0	− ·55	− ·493	− ·727	− ·957	−1·18	−1·39
$t =$	·001713	·00204	·00240	·00280	·00325	·00376	·00435
$i_1 =$	3·5	4·0	4·5	5·0	5·5	6·0	6·5
$i_2 =$	−1·59	−1·77	−1 94	2·09	−2·22	−2·33	−2·41
$t =$	·00504	·00588	·00694	·00386	·01047	·01423	—
$i_1 =$	7·0	7·5	8·0	8·5	9·0	9·5	10
$i_2 =$	−2·44	−2·43	−2·37	−2 20	−1·87	−1·27	—

Using this table, we may plot the curves showing the growth of the current i_1 to its steady value of 10 amperes, and the growth and decay of the current i_2. These curves are shown in Fig. 75. The student will find it instructive to compare Fig. 75 with Fig. 26 (Art. 36), which shows the rise of current in the circuit A when there is no mutual inductance. For the sake of convenience, the curve of Fig. 26 is reproduced (dotted) in Fig. 75. The student will at once notice that one of the effects of mutual inductance is to *accelerate the initial growth of the current.* Starting from the origin, the two curves *diverge*, then for a time run almost parallel to each other; after that they begin to *converge*, and ultimately the dotted curve *crosses* the full line one, so that at a certain instant (corresponding to a considerable fraction of the steady value of the current) the current will have the same value, whether mutual inductance be present or absent. Beyond this point the current will grow more rapidly if mutual inductance is absent.

107. The above results might have been predicted from the following purely physical considerations. When an E.M.F. is applied to the circuit A (Fig. 74), then, as soon as a current makes its appearance in the circuit, it induces a current in the circuit B, the direction of this induced current being that of the dotted arrow. The induced current in B reacts on A, and induces in A an *adjuvant* E.M.F., thus accelerating the growth of the current. This goes on so long as the current in B is increasing numerically. A stage, however, is soon reached when the E.M.F. induced in B is too small to maintain the current, and the latter begins to decrease. During the decrease of the current in B an *opposing* E.M.F. is induced in A, which now *retards* the growth of the current. This fully explains the crossing of the two curves, and we see why it is that the growth of the current in a circuit is during its early stages *accelerated*, and during its later stages *retarded*, by the presence of mutual inductance.

108. Consider a circuit made up of a conductor consisting of two hollow coaxial cylindric shells s_1 and s_2, as shown in longitudinal and transverse sections in Fig. 76. We may imagine the cylindric shells to be two thin tubes of copper placed coaxially

and bent into the shape shown, the space between the tubes being filled with some insulating material which serves to keep them apart. Let the ends of the tubes or shells be connected by means of stout copper plates P_1 and P_2 soldered to them, and let a steady E.M.F. be introduced into the circuit so formed.

We have here the case of two circuits, s_1 and s_2, joined in parallel to a source of steady E.M.F. Let us investigate the law of growth of the current in each of the two circuits.

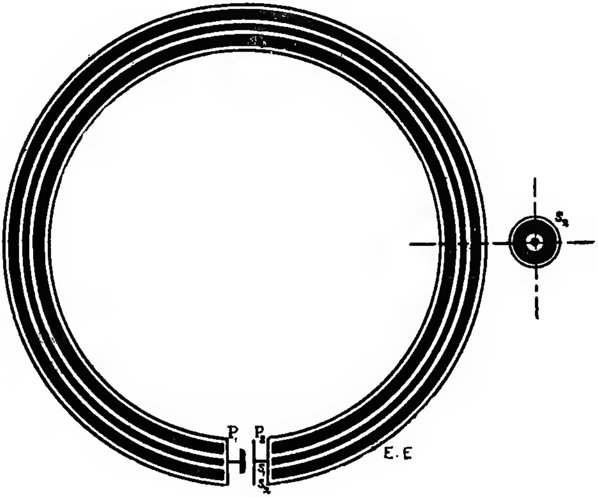

FIG. 76.

For the sake of simplicity, we shall suppose the cross-sections of the shells to be the same, so that both shells have the same resistance, say r. The magnetic field due to the current in s_1 is entirely confined to the region outside the inner boundary of this shell; similarly, the magnetic field due to the current in the outer shell s_2 is entirely confined to the region outside the inner boundary of that shell. From this it is at once obvious that the inductance L_1 of s_1 is greater than L_2, that of s_2; for a unit current in either shell will produce the same magnetic flux outside the shells; and besides this flux, the current in s_1 produces a magnetic flux in the space between the two shells, whereas the current in s_2 produces no such flux. It is also at once obvious that the whole of the flux through the circuit s_2 must pass through the circuit s_1; in other words, the mutual inductance M of the two shells is equal to L_2, the inductance of the outer shell.

Since the two shells have the same resistance, but the outer shell a smaller inductance, we should expect (Art. 34) the rise of current to its steady value to be more rapid in this outer shell. Besides the effect of (self) inductance, however, we have also to consider that produced by the mutual inductance of the two shells.

Let E denote the steady E.M.F. simultaneously applied to the two shells. In each shell we have three E.M.F.'s, the sum of which must equal E. These three E.M.F.'s are:

(1) The $r\,i$ E.M.F., due to the resistance of each shell.

(2) The induced E.M.F. in each shell due to the changing magnetic flux produced by *its own* current.

(3) The induced E.M.F. in each shell due to the variation of magnetic flux caused by the current in *the other* shell.

If i_1 and i_2 stand for the instantaneous values of the currents in the shells s_1 and s_2 respectively, i_1' and i_2' for their rates of change, then we have the equations:

$$r\,i_1 + L_1\,i_1' + M\,i_2' = E\,;$$
$$r\,i_2 + L_2\,i_2' + M\,i_1' = E\,;$$

which now take the place of the equations (α) and (β) of Art. 105. Now, since $L_2 = M$, we may insert this value for L_2 in the second equation. Subtracting the second equation from the first, we obtain

$$i_1' = \frac{r\,(i_2 - i_1)}{L_1 - M}.$$

Again, multiplying the first equation by M, the second by L_1, and then subtracting the second from the first, we find

$$i_2' = \frac{E}{M} - \frac{r\,(L_1\,i_2 - M\,i_1)}{M\,(L_1 - M)}.$$

Using these equations and applying the step-by-step process explained in Art. 106, we may plot the curves showing the rise of current in each of the two shells.

109. The expressions obtained for i_1' and i_2' at once indicate a remarkable fact. At the commencement, when the time $t = 0$, and $i_1 = 0$, $i_2 = 0$, we find $i_1' = 0$, $i_2' = \frac{E}{M}$. In other words, while *the current in the outer shell* s_2 *at once begins to rise at a rapid rate* $\left(\frac{E}{M}\right)$, *the current in the inner shell does not begin to rise until the current in* s_2 *has reached a sensible value.* The curve showing the rise of current in the outer shell will make a certain angle with the axis of time at

the origin, whereas the curve for the inner shell will be *tangential* to the axis of time, as shown in Fig. 77.

110. We shall apply these results to a numerical example. Let $r = \cdot 001$; $L_1 = 2\cdot 5 \times 10^{-6}$; $L_2 = M - 2 \times 10^{-6}$, and let $E = \cdot 1$. We then find

$$i_1' = 2{,}000\ (i_2 - i_1),$$
$$i_2' = 5 \times 10^4 - 1{,}000\ (2\cdot 5\ i_2 - 2\ i_1).$$

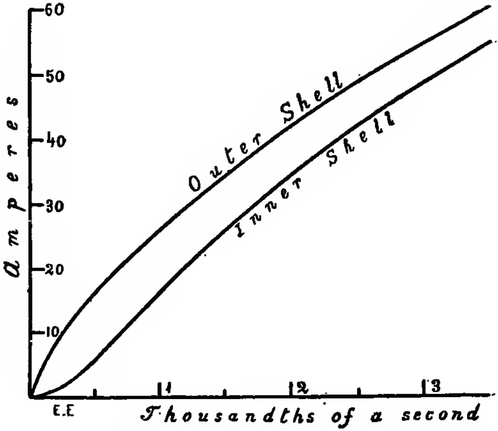

FIG. 77.

Using the step-by-step process already explained (Art. 106), we obtain the following set of values:

t =	0	10^{-4}	$2\cdot 33 \times 10^{-4}$	$4\cdot 57 \times 10^{-4}$	$6\cdot 75 \times 10^{-4}$	$9\cdot 4 \times 10^{-4}$
i_1 =	0	0	1·33	5·21	9·47	15
i_2 =	0	5	10	15	20	25
t =	$12\ 26 \times 10^{-4}$		$15\cdot 27 \times 10^{-4}$	$18\cdot 58 \times 10^{-4}$	$22\cdot 1 \times 10^{-4}$	26×10^{-4}
i_1 =	20·8		26·3	32·1	37·7	43·3
i_2 =	30		35	40	45	50

These results are plotted in Fig. 77. The student will see that the rise of current is much more rapid in the outer shell.

111. Instead of two coaxial cylindric shells, let us next consider a *solid* cylindric conductor. Imagine the conductor split up into a large number of thin cylindric shells. The results deduced above may be applied to each pair of shells, and we are thus led to the conclusion that when a P.D. is established between the two ends of a cylindric conductor, the current rises in the surface layers of the conductor first, and only when it has attained an appreciable value in these surface layers does it begin to penetrate deeper into the conductor.

Similar considerations hold good for a conductor of any shape. The current always starts from the *surface* layers and gradually becomes diffused over the whole cross-section of the conductor.

Notes on Chapter XIII.

The student may be helped towards a clearer understanding of the phenomena of mutual inductance by referring to the description of an ingenious model designed by Clerk Maxwell, given at the end of Chapter VII., Vol. II., of his "Electricity and Magnetism" (third edition, p. 228). The induction of transient currents in a closed circuit placed in the neighbourhood of another circuit in which a current could be started or arrested was first studied by Faraday in England and by Joseph Henry in America.

EXERCISES.

1. The mutual inductance of two circuits is ·02 henry. One of the circuits, whose resistance is 5 ohms, is connected in series with a ballistic galvanometer of resistance = 20 ohms. A current of 5 amperes is established in the other circuit. Find the quantity, in microcoulombs, which will pass through the ballistic galvanometer.

2. It is desired to find the mutual inductance of two circuits. One

of them, of resistance = 2 ohms, is joined in series with a ballistic galvanometer of 20 ohms resistance, and a current of 1 ampere is established in the other circuit. On breaking this circuit, it is found that the galvanometer deflection corresponds to 46 microcoulombs. What is the mutual inductance of the two circuits?

3. Two coils, one of which conveys a current of 2 amperes and the other of 3 amperes, are placed with their planes parallel to each other. Their mutual inductance in this position is ·006 henry. One of the coils is then rotated through 90deg., so that its plane becomes normal to that of the other coil. If the currents in the two coils are maintained constant during the motion, what is the energy, in ergs, spent in rotating the coil (mechanical friction neglected)?

4. The resistance of a certain coil is 8 ohms, its inductance ·08 henry; another coil, of resistance = 1 ohm and inductance = ·05 henry, is placed in such a position relatively to the first coil that their mutual inductance is ·03 henry. The second coil is short-circuited, and a steady E.M.F. of 32 volts is applied to the first coil. Draw the curves showing the rise of current in the first coil, and the rise and decay of current in the second coil.

CHAPTER XIV.

112. We next proceed to consider the problem of two circuits having mutual inductance, a simple harmonic impressed E.M.F. acting on one of the circuits.

Let r_1, L_1 and r_2, L_2 denote the resistance and inductance respectively of each circuit. We shall assume that the currents i_1 and i_2 in the two circuits follow the simple harmonic law, and that the current i_2 lags behind i_1, so that we may write

$$i_1 = I_1 \sin p\,t,$$
$$i_2 = I_2 \sin\,(p\,t - \theta).$$

If we suppose that e stands for the instantaneous impressed E.M.F. in the first circuit, then, by the principles explained in Art. 105, the currents must satisfy the equations

$$\left.\begin{aligned} r_1\,i_1 + L_1\,i_1' + M\,i_2' &= e \\ r_2\,i_2 + L_2\,i_2' + M\,i_1' &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad (\epsilon)$$

where M is the mutual inductance of the two circuits.

Since we suppose i_1 and i_2 to follow the simple sine law, i_1' and i_2' will, by Art. 22, follow the same law, and their amplitudes will be $p\,I_1$ and $p\,I_2$ respectively. If we draw two rotating vectors, O A and O B (Fig. 78a), of lengths I_1 and I_2 respectively, and making an angle θ with each other, then, supposing the angular velocity in a counter-clockwise direction to be p, the projections of these rotating vectors on the vertical axis will give the instantaneous values of i_1 and i_2. The projection of a rotating vector O C, of length $r\,I_1$, will give the instantaneous value of the $r\,i_1$ component of the impressed E.M.F. The projections of two other vectors, O D and O E, of lengths $L_1 p I_1$ and $M p I_2$, and drawn at right angles to O A and O B respectively (Art. 22), will give the instantaneous values of the $L_1 i_1'$ and $M i_2'$ components of the E.M.F. respectively. By the first of the equations (ϵ) above, the sum of the projections of O C, O D, and O E on the vertical axis at any instant must give the value of the impressed E.M.F. at that instant. The amplitude of the impressed E.M.F. is therefore represented by the length of the rotating vector which is the resultant of the three vectors O D, O C, and O E.

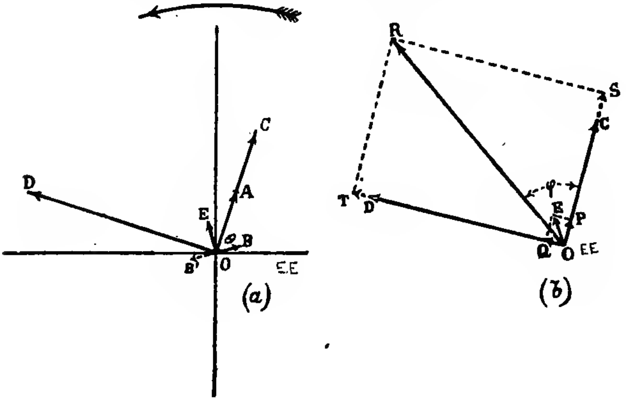

FIG. 78.

In order to find this resultant, we replace the vector O E by two components (Art. 11), as shown in

Fig. 78*b*, one of which, O P, is along O C, while the other, O Q, is along O D. By Art. 12, we have

$$\text{O P} = \text{O E} \cos \text{P O E} = \text{M} p \, \text{I}_2 \sin \theta,$$

and $$\text{O Q} = \text{O E} \cos \text{Q O E} = \text{M} p \, \text{I}_2 \cos \theta.$$

Adding O C and O P, we obtain the vector O S; adding O D and O Q, we obtain O T, where

$$\text{O S} = r \, \text{I}_1 + \text{M} p \, \text{I}_2 \sin \theta,$$
$$\text{O T} = \text{L} p \, \text{I}_1 + \text{M} p \, \text{I}_2 \cos \theta.$$

Finally, compounding O S and O T, we obtain

$$\text{O R} = \sqrt{\text{O S}^2 + \text{O T}^2}$$
$$= \sqrt{(r \, \text{I}_1 + \text{M} p \, \text{I}_2 \sin \theta)^2 + (\text{L} p \, \text{I}_1 + \text{M} p \text{I}_2 \cos \theta)^2} \quad . \quad (l)$$

We have thus found the amplitude O R of the impressed E.M.F. in terms of I_1, I_2, and θ.

We next construct a vector diagram for the second circuit. This is shown in Fig. 79, in which $\text{O L} = r_2 \, \text{I}_2$, $\text{O M} = \text{L}_2 p \, \text{I}_2$, $\text{O N} = \text{M} p \, \text{I}_1$; O M and O N being at right angles to O B and O A respectively. By the second of the equations (ϵ) above, the sum of the projections of O L, O M, and O N on the vertical axis at any instant must vanish. Hence the resultant of the vectors O L, O M, and O N must equal nothing. Proceeding as in Fig. 78*b*, we find that

$$\text{O N} \cos \theta + \text{O M} = 0,$$

and $$\text{O N} \sin \theta = \text{O L};$$

or, since $\text{O L} = r_2 \text{I}_2$, $\text{O M} = \text{L}_2 p \, \text{I}_2$, and $\text{O N} = \text{M} p \, \text{I}_1$,

$$\text{M} p \, \text{I}_1 \cos \theta = - \text{L}_2 p \, \text{I}_2 \quad . \quad . \quad . \quad (m)$$
$$\text{M} p \, \text{I}_1 \sin \theta = r_2 \text{I}_2 \quad . \quad . \quad . \quad . \quad (n)$$

Squaring these two equations, and adding, we get

$$\text{I}_2 = \frac{\text{M} p \, \text{I}_1}{\sqrt{r_2^2 + p^2 \, \text{L}_2^2}} \quad . \quad . \quad . \quad . \quad (o)$$

Again, dividing the second equation by the first, we find

$$\tan\theta = -\frac{L_2 p}{r_2}.$$

The angle of lag θ of the current i_2 behind the current i_1 is therefore greater than π (since its sine is positive and its cosine negative), and the actual position of the rotating vector O B is as shown by the dotted line O B′ in Fig. 78.

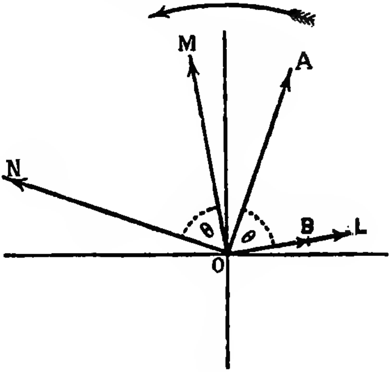

FIG. 79.

113. Using equations (l), (m), (n), and (o) of the preceding article, we may express everything in terms of the amplitude E (= O R, Fig. 78b) of the impressed E.M.F., and the resistances and inductances of the two circuits. We thus find

$$I_1 = \frac{E}{\sqrt{\left(r_1 + \frac{r_2 p^2 M^2}{r_2^2 + p^2 L_2^2}\right)^2 + p^2\left(L_1 - \frac{M^2 p^2 L_2}{r_2^2 + p^2 L_2^2}\right)^2}} \quad (s)$$

Again, if ϕ denote the angle by which i_1 lags behind the impressed E.M.F., we see from Fig. 78*b* that

$$\tan \phi = \frac{p\left(L_1 - \frac{M^2 p^2 L_2}{r_2^2 + p^2 L_2^2}\right)}{r_1 + \frac{r_2 p^2 M^2}{r_2^2 + p^2 L_2^2}} \quad . \quad . \quad . \quad (t)$$

From equations (*s*) and (*t*) it is evident that if, while a simple harmonic E.M.F. of amplitude E is applied to a circuit of resistance r_1 and inductance L_1, another circuit, of resistance r_2 and inductance L_2, be brought near the first circuit and placed in such a position that their mutual inductance is M, then the first circuit behaves as if it had a resistance

$$r_1 + \frac{r_2 p^2 M^2}{r_2^2 + p^2 L_2^2},$$

and an inductance

$$L_1 - \frac{M^2 p^2 L_2}{r_2^2 + p^2 L_2^2}.$$

In other words, the apparent resistance of the first circuit is increased, and its apparent inductance diminished, by the presence of another closed circuit in its neighbourhood.

114. We shall now consider the case of a circuit similar to that described in Art. 108, and suppose that a simple harmonic E.M.F. is applied to it. We may imagine the circuit to be made up of a length of concentric main, consisting of two tubular copper conductors, s_1 and s_2 (Fig. 80), separated by some dielectric. Instead, however, of using one of the tubular conductors as the forward, and the other as

the return main, we shall suppose the outer and inner shells joined together at their ends, and connected to a source of simple harmonic E.M.F. We thus have, as in Art. 108, the case of two circuits the mutual inductance of which is equal to the (self) inductance of one of them.

Let r be the resistance of each circuit (the two shells being supposed to have the same cross-sectional area); L_1 the inductance of the inner shell, and M the inductance of the outer shell; M will also be equal to the mutual inductance of the two shells. Further, let i_1 and i_2 stand for the instantaneous currents in the inner and outer shells respectively, and let e be the instantaneous impressed E.M.F. By the principles explained in Art. 105, the equations of the currents in the two shells are

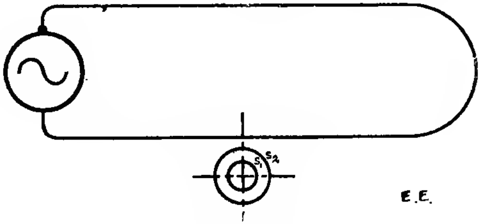

FIG. 80.

$$r i_1 + L_1 i_1' + M i_2' = e,$$
$$r i_2 + M i_2' + M i_1' = e.$$

Subtracting the second equation from the first, we find

$$r (i_1 - i_2) + (L_1 - M) i_1' = 0,$$

whence $$i_2 = i_1 + \frac{L_1 - M}{r} i_1'.$$

115. The two currents being simple harmonic ones, we may represent i_1 by the projection on the vertical axis of a rotating vector O A (Fig. 81) of length I_1 (where I_1 is the amplitude of i_1). The rate of change of i_1—*i.e.*, i_1'—would be represented (Art. 22) by the

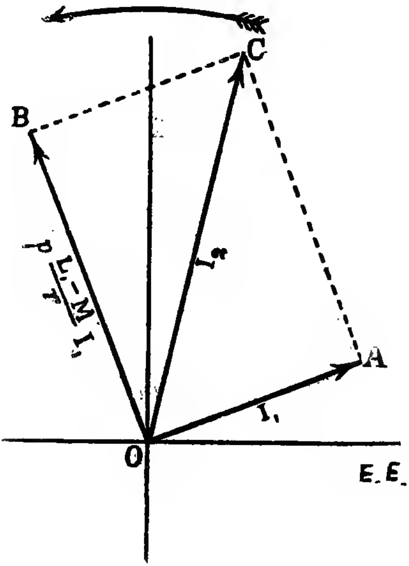

FIG. 81.

projection of a rotating vector of length $p\,I_1$, drawn at right angles to O A. Hence the term $\frac{L_1-M}{r}\,i_1'$, which is the second term on the right-hand side of the last equation in the preceding article, may be represented by the projection on the vertical axis of a rotating vector O B, of length $\frac{p\,(L_1 - M)}{r} I_1$. By the equation of last article, i_2 is equal to the sum of the projections of O A and O B on the vertical axis—*i.e.*, to the projection of O C, which is the resultant of O A and O B. Now if we denote the amplitude of i_2 by I_2, then

$$I_2 = \mathrm{O\,C} = \sqrt{\mathrm{O\,A}^2 + \mathrm{O\,B}^2}$$

$$= I_1 \sqrt{1 + \frac{p^2\,(L_1 - M)^2}{r^2}}$$

so that

$$\frac{I_2}{I_1} = \sqrt{1 + \left\{\frac{p\,(L_1 - M)}{r}\right\}^2}.$$

Hence the current in the outer shell is always greater than that in the inner shell. Further, a glance at Fig. 81 shows that *the current in the inner shell lags behind that in the outer shell.*

As an example, let us suppose that $r = {\cdot}1$ ohm, $L_1 = {\cdot}0015$ henry, $M = {\cdot}001$ henry, and let the periodicity be 100 ∿ per second. Then $p = 2\pi \times 100 = 628$ approximately, so that

$$\frac{I_2}{I_1} = \sqrt{1 + \left(\frac{628 \times {\cdot}0005}{{\cdot}1}\right)^2};$$

$$= \sqrt{1 + 9{\cdot}86};$$

$$= 3{\cdot}29.$$

In this case, therefore, the current in the outer shell is more than three times as great as in the inner shell.

One effect of this crowding of the current towards the outer shell is to increase the rate at which energy is being dissipated, so that the resistance of the two shells joined in parallel to an alternating current is very much greater than their resistance to a continuous current. In order to compare the two resistances, let us suppose that a continuous current of i amperes is sent through the two shells. The resistance of each shell being ·1 ohm, their joint resistance (to a continuous current) is ·05 ohm. The rate at which energy is being dissipated is therefore

$$\cdot 05\, i^2.$$

Next, let an alternating current of i amperes be sent through the shells; of this, $\frac{i}{4\cdot 29}$ amperes will pass through the inner shell, and $\frac{i \times 3\cdot 29}{4\cdot 29}$ amperes through the outer shell. The total rate of dissipation of energy with an alternating current of i amperes is therefore

$$\cdot 1 \times \frac{i^2}{4\cdot 29^2} + \cdot 1 \frac{i^2 \times 3\cdot 29^2}{4\cdot 29^2} = \cdot 0643\, i^2;$$

i.e., the combination of the two shells joined in parallel behaves as if its resistance to an alternating current were ·0643 ohm instead of ·05 ohm (the resistance offered to a continuous current).

116. The equation

$$\frac{I_2}{I_1} = \sqrt{1 + \left\{\frac{p(L_1 - M)}{r}\right\}^2}$$

indicates two more facts which are of considerable interest and importance.

In the first place, we see that for given diameters and thicknesses of shells the ratio of the two currents will depend on the resistivity (specific resistance) of the material used. The lower the resistivity the smaller will be r, and hence shells made of a highly conducting material will exhibit the effect better than those made of a poorly conducting substance.

In the next place, we see that the ratio of the two currents will depend on the magnetic permeability of the medium separating the two shells. We have supposed the permeability to be unity. If, however, we imagine an iron shell interposed between the two shells conveying the current, the iron shell being insulated, then it is obvious that the value of $L_1 - M$ will be enormously increased, owing to the high permeability of the iron. The mere interposition of a highly permeable substance will therefore greatly increase the ratio of I_2 to I_1. The iron shell acts as a sort of screen to the inner shell, diverting a large portion of the current from the inner to the outer shell.

117. It is, of course, at once obvious that the ratio $\frac{I_2}{I_1}$ will depend on the periodicity. As the periodicity is increased indefinitely, this ratio increases indefinitely. For very high periodicities, the term $\frac{p(L_1 - M)}{r}$ will be very great compared with unity, so that we may write approximately

$$\frac{I_2}{I_1} = p\,\frac{L_1 - M}{r},$$

and we see that the ratio of the two currents, for very high periodicities, is in direct proportion to the periodicity.

118. Instead of considering two thin cylindric shells, we may next take the case of a solid cylindric conductor. Imagine the conductor split up into a large number of thin concentric cylindric shells. If we take any two shells, then, from the results deduced above, it follows that the outer shell will carry more current than the inner one. Each shell will, in fact, be subjected to the screening action of all the shells lying outside it. Hence the current will be unequally distributed over the cross-section of the conductor, being crowded towards the outer portions of it. This crowding of the current towards the surface of the conductor produces a twofold effect: *it increases the resistance and diminishes the inductance* of the conductor. For let us consider two equal cylindric conductors conveying the same total current, and let the current density be uniform over the cross-section of the first conductor, while it increases from the centre outwards in the second conductor. Consider two cylindric shells, s_1 and s_2 (Fig. 82), of equal

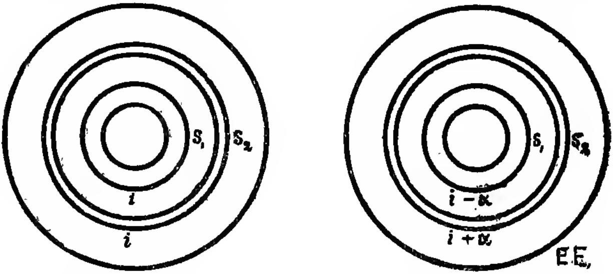

Fig. 82.

resistance r, in the two conductors. Let i denote the current passing through each shell in the first conductor, and let $i - a$ and $i + a$ denote the currents in s_1 and s_2 respectively in the second conductor. The total rate of dissipation of energy in s_1 and s_2 is

in the first conductor, $2\,r\,i^2$,

and in the second, $r\,(i - a)^2 + r\,(i + a)^2 = 2\,r\,i^2 + 2\,r\,a^2$.

Hence the second conductor behaves as if it had a greater resistance than the first.

Again, by transferring a current a from the inner shell s_1 to the outer shell s_2, we diminish the magnetic flux due to that current, for the whole of the lines *between the two shells* which were due to the current a while it was flowing in s_1 have disappeared on its transference to the shell s_2. We thus see that a crowding of the current towards the outer portions of the conductor results in a reduction of the total magnetic flux due to the current—*i.e.*, the inductance of the conductor is thereby diminished.

119. The unequal distribution of an alternating current over the cross-section of a conductor is a matter of considerable importance from a practical point of view. With conductors of large size this effect becomes so serious that the inner portions of the conductor practically contribute nothing towards its current-carrying capacity. Hence, conductors intended to carry large alternating currents frequently take the form of tubes or flat strips. It may be mentioned that for ordinary periodicities, ranging from, say, 80 to about 150 ∼ per second, this "skin effect" does not become appreciable (for copper conductors) until a

diameter of about ½in. has been reached. Beyond that, the effective resistance of a conductor to an alternating current appreciably exceeds its resistance to a continuous current, and the increase of resistance increases rapidly with the diameter. Thus, a copper conductor ·75in. in diameter offers about 17 per cent. more resistance to an alternating current of periodicity 130 ∿ per second than it does to a continuous current.

In the case of *iron* conductors the "skin effect" is enormously magnified owing to the high permeability of the material (Art. 116), so that even with comparatively thin wires the effect is appreciable at ordinary periodicities.

For currents alternating with very great rapidity, such as the oscillatory discharges of Leyden jars, or oscillatory lightning discharges, the only effective portion of a conductor is an extremely thin surface layer. If it is desired to reduce the resistance as much as possible, the shape of the conductor must be so chosen as to offer a large *surface*. A thin flat ribbon, for example, would offer much less resistance to a rapidly varying current than a cylinder of equal cross-section.

Besides using tubular or flat conductors, we may further ensure a uniform distribution of current over the cross-section by stranding the conductor, each strand being *insulated* from the others, and the stranding being carried out in such a manner that the portion of each strand which lies on the surface of the conductor is about the same for all the strands. This is the method of stranding adopted in Lord Kelvin's current balances intended for the measurement of large alternating currents.

120. Closely allied to this subject is the unequal distribution of magnetic induction over the cross-section of a mass of iron which is subjected to an alternating magnetic force, and this effect we now proceed to consider. The problem is very much simpler and more easily understood than the one we have just investigated.

Let us suppose that a mass of iron is suddenly subjected to the action of a steady magnetic force. During the growth of the magnetic induction, currents will be induced in the mass of iron, the direction of these currents being such that they tend to oppose the growing magnetic induction. Each current-sheet will exert a demagnetising effect at all points lying within its boundary. Hence the demagnetising effect produced by the currents will increase as we proceed from the surface of the iron towards its interior portions. We thus see that the magnetic induction will propagate itself from the surface inwards, just as a current diffuses itself from the surface of a conductor towards its central portions when a steady E.M.F. is applied to the conductor.

If, for example, we take the case of a cylindrical rod of iron, shown in section in Fig. 83, and suppose that a steady magnetic force, having a direction from front to back of the plane of the paper, is suddenly applied to it, then the growing magnetic induction will give rise to cylindrical current-sheets, having the direction indicated in Fig. 83. The demagnetising effect produced by these current-sheets will obviously increase as we proceed from the surface towards the centre of the rod. Hence, the magnetic induction will reach its steady value in the surface layers first, and then gradually diffuse itself over the whole cross-section of the rod.

Instead of supposing the magnetic force to remain constant, let us assume that it varies according to the simple harmonic law. The varying magnetic induction* in the iron will give rise to alternating-current

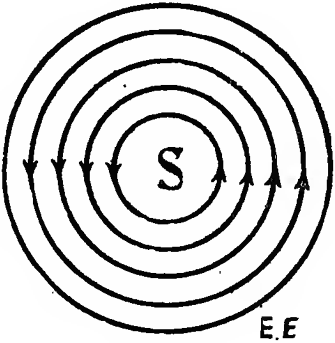

FIG. 83.

sheets which will exert a demagnetising effect on the central portion of the iron core. With a very rapid rate of alternation the induction will be practically confined to the surface layers of the iron. This is the analogue of the "skin effect" obtained with a conductor conveying a rapidly alternating current, and it has an important bearing on transformer design.

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* The magnetic induction will *not* vary according to the simple harmonic law, but will follow one much more complex.

Notes on Chapter XIV.

The method explained (in an elementary manner) in the present chapter of dealing with the problem of two circuits possessing mutual inductance was first introduced by Clerk Maxwell. It is applicable to a transformer with a core of non-magnetic material.

A very simple approximate method of dealing with the problem of a transformer was introduced at a later date by Dr. Hopkinson, after transformers had come into use for electric lighting purposes. In a complete theory of the action of a transformer, the *hysteresis* of the core must be taken into account. The student possessing the necessary mathematical knowledge will find this theory fully set forth in Dr. Fleming's book, "The Alternate-Current Transformer," already referred to several times. A clear account of the principles underlying transformer design will be found in "The Design of Alternate-Current Transformers," by Mr. R. W. Weekes (Biggs and Co.), or in "Transformers," by Mr. G. Kapp (Whittaker and Co.).

Clerk Maxwell appears to have been the first to point out that an alternating current does not distribute itself uniformly over the cross-section of a conductor. Mr. Oliver Heaviside, Lord Rayleigh, and Lord Kelvin have dealt with the problem mathematically, and Prof. Hughes devised some striking experiments to verify the deductions of theory.

The effect of eddy currents in iron plates in crowding the magnetic induction towards the surface layers of the plate has been fully worked out by Prof. J. J. Thomson. Dr. Hopkinson and Mr. Wilson have recently investigated experimentally the propagation of magnetic induction in solid iron cylinders, as affected by the eddy currents induced in them.

EXERCISES.

1. The mutual inductance of two coils is ·03 henry. An alternating current of 1 ampere, at a periodicity of 100 ∿ per sec., is sent through one of the coils. What will be the reading of an electrostatic voltmeter connected across the terminals of the other coil?

2. The resistances of two coils are 4 and 3 ohms, their inductances ·02 and ·03 henry respectively, and their mutual inductance is ·01 henry. Find the current which will flow in the second coil when it is short-circuited, and a current of 2 amperes, at a periodicity of 80 ∿ per sec., is sent through the first coil.

3. The mutual inductance of two coils is ·01 henry. One of them has a resistance of 3·34 ohms, and it is found that a current of

5 amperes will pass through it when the P.D. is 50 volts, and the circuit of the second coil open. If the P.D. remain unaltered, and the second coil, of resistance = 5 ohms and inductance = ·02 henry, be short-circuited, what will be the value of the current through the first coil, the periodicity being 100 ⁓ per sec. ?

CHAPTER XV.

121. In the next few articles we shall briefly consider the behaviour of ordinary alternators when used as motors, and the parallel running of alternators.

Let the thick-line rectangle in Fig. 84*a* represent one of the coils of an alternator armature, in the position of maximum E.M.F., the direction of motion and that of the induced E.M.F. being indicated by the arrows. Suppose, for the sake of simplicity, that the current is in phase with the E.M.F. Then the coil in the position *a* experiences the maximum pull, the pull being opposed to the direction in which the coil is being driven by the engine. The pull gradually diminishes as the current dies down to nothing, and disappears entirely when the coil reaches the position *b*. As the coil passes from this latter position to the position *c*, the current induced in it is in the opposite direction, but since the coil is approaching a pole of opposite polarity, there will be an opposing pull which gradually increases, reaching a maximum value for the position *c*. The pull will then once more diminish, and so on.

We thus see that when the current is in phase with the E.M.F., the alternator armature experiences a retarding torque, which varies periodically with the current, rising from nothing to a certain maximum value, and then dying down to nothing again. The direction of this torque, however, is always such as to oppose the driving torque. The resistance offered by a loaded alternator (always supposing the current to be in phase with the E.M.F.) to the engine driving it is therefore not a constant but a rapidly fluctuating one, resembling in this respect the resistance offered

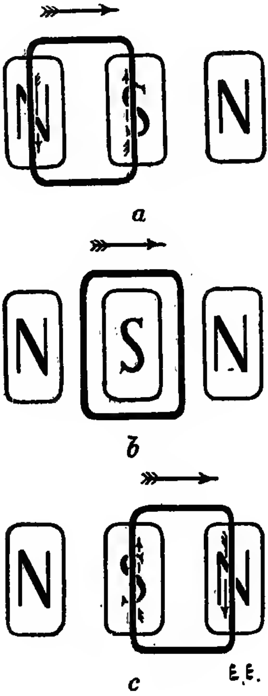

FIG. 84.

by a circular saw cutting through a thin sheet of metal.

122. While the alternator is running, suppose the

belt driving it to be suddenly thrown off, and an alternating current to be sent through the armature coils which has the same periodicity and amplitude as that developed by the alternator whilst it was being driven by the engine, but which is at every instant in opposite phase. Also, suppose that at the same time a brake is applied to the alternator pulley, so that the total resisting torque is the same as the mean resisting torque which was formerly due to the current circulating in the armature coils. The direction of the current for the positions *a* and *c* in Fig. 84 will now be the reverse of that indicated by the arrows, so that what was formerly an opposing torque will now become a driving torque. The driving torque will undergo precisely the same rapid fluctuations as those which characterised the resisting torque when the machine was being driven. The resisting and driving torques being equal, it is obvious that the alternator will go on running at the same speed as before, thus acting as a *motor*.

123. Such a motor possesses the peculiarity that it will run at only one definite speed, depending on the periodicity of the alternating current which drives it. For this reason, it is termed a *synchronous* motor. If we suppose that by a very sudden and great increase of load the speed of the motor is reduced, then it is obvious that what was formerly a regular succession of driving impulses will now become a perfectly irregular series of pulls on the alternator coils, now acting in one direction, now in the opposite, and with their algebraical sum over any considerable time interval equal to practically nothing; for now

the direction of the current bears no definite relation to the position of the armature coils with respect to the poles. The motor will therefore quickly come to a standstill.

124. In order to start such a motor, it becomes necessary to run it up to its *speed of synchronism*[*] by some independent means, such as a small auxiliary motor. Further, the *phase* of the E.M.F. of the motor must be adjusted so as to be in opposition to the phase of the generator driving it before the switch is closed.[†] These operations are performed by using an arrangement known as a *synchroniser*. The form of synchroniser commonly used (Fig. 85) consists of two transformers, T_1 and T_2, with their primaries connected across the terminals of the generator G and the motor M respectively, while their secondaries may be closed through a voltmeter V (or incandescent lamp), the connections being so arranged that when the two machines are in opposition as regards phase, the E.M.F.'s in the secondaries of the synchronising transformers are added together, and the voltmeter shows the highest reading. If the speeds of the two machines are different, the voltmeter reading will show fluctuations; the switch S must be closed at the instant when the voltmeter gives the maximum reading, and when that reading is steady.

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[*] The speed of synchronism of a motor depends, for a given periodicity, on the number of poles in the motor field.

[†] Otherwise there would be an enormous current through the motor at the instant of switching on, due to the fact that the motor E.M.F. is *added* to that of the generator. This current would tend to stop the motion of the motor, and the shock might be sufficient to rip out the coils.

125. We have hitherto supposed that the current through the motor armature is exactly opposed in phase to the E.M.F. Under these circumstances the current passes through its zero value at the instant when the coil is facing a pole,* and is therefore in the best position for exerting an effect on the field.

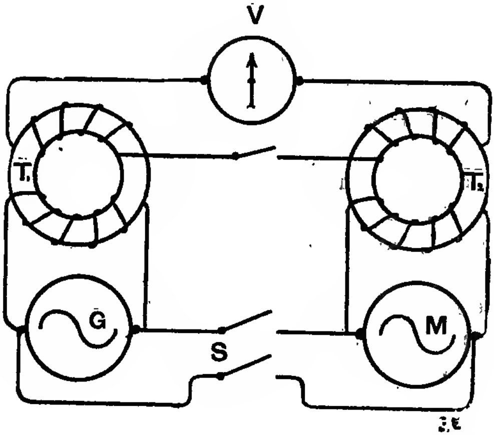

FIG. 85.

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* If we suppose that the cross-magnetising or distorting effect of the armature current on the field is negligible. In a continuous-current dynamo, the armature, as is well known, distorts the field, twisting it round in the direction of rotation. A similar action takes place in alternators; but instead of a steady cross-magnetising force, we here have to deal with a series of impulses. These impulses do not, however, produce variations of corresponding amplitude in the field (unless the latter is finely laminated throughout—see Art. 120). The field is thus permanently distorted, although the distortion does not remain constant, but fluctuates with the armature current within certain narrow limits. From this it follows that when a coil has come opposite a pole, the E.M.F. in it has not quite died down to zero, for its leading side is moving in a somewhat more intense field than its trailing side. In what follows, we neglect this effect, for the sake of greater simplicity.

With the current exactly opposed in phase to the E.M.F. we thus see that there will be no tendency on the part of the armature to either weaken or strengthen the field. It has already been shown (Art. 78) that in the case of a generator the armature current tends to *weaken* the field if the zero value of the E.M.F. occurs *before* the zero value of the current, and to *strengthen* it if it occurs *after* the zero value of the current. The opposite effect takes place in a motor, as will be readily seen by an inspection of Fig. 86. In Fig. 86*a*, the E.M.F. passes through its zero value *before* the

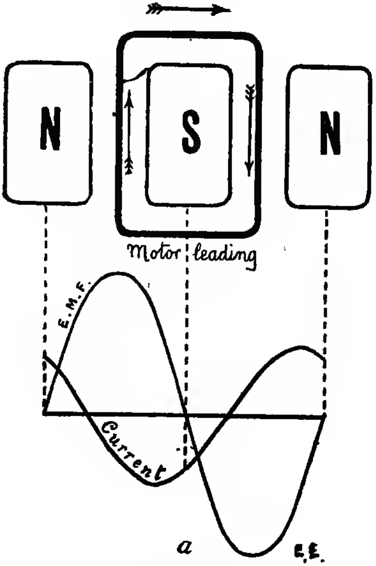

FIG. 86*a*.

current, and the armature tends to *strengthen* the field; in Fig. 86*b* it passes through its zero value *after* the current, and the field is weakened.

It must be borne in mind that, owing to the rapid pulsations of the current in the armature coils, the magnetising force due to the armature is not a steady one, but consists of a series of impulses following each other in rapid succession. The effect produced by the armature current might be imitated by winding a number of turns of wire on the field magnet, and supplying these with current impulses corresponding to those due to the armature. These current impulses

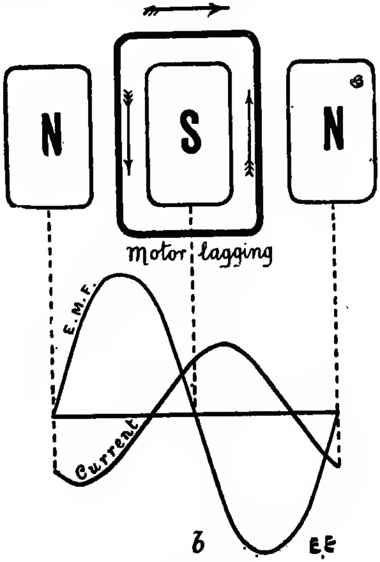

FIG. 86*b*.

would not, however, produce variations of corresponding amplitude in the magnetic flux through the field magnets. A mass of iron, as we have seen (Art. 120), does not immediately respond to a magnetic force (unless very finely laminated).* The result is that the magnetic flux through the field does not follow the very rapid variations of current, but assumes a certain intermediate state which is practically steady. We are thus led to the conclusion that if the E.M.F. in a synchronous motor passes through its zero value *before* the current the field is permanently *strengthened*, while if it does so *after* the current, the field is permanently *weakened*. In other words, the E.M.F. in the first instance is increased, and in the second diminished, by the *armature reaction*.

126. Besides the weakening or strengthening of the field by the armature current when the latter is not exactly opposite in phase to the E.M.F., there are other actions which we have to consider. Referring to Fig. 86*a*, we notice that when the coil moves past the position there indicated, the force acting on it will be a *retarding* one, tending to arrest the motion of the armature until the current becomes reversed. Similarly, in Fig. 86*b*, the coil will, from the instant of reversal of current until it reaches the position shown, be acted on by a retarding force. In each case, therefore, the impulsive torques driving the motor will be alternately positive and negative, and the useful driving torque will be the difference between a positive and a negative one. The motor will in

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* This, in fact, renders possible the excitation of the field magnets of an alternator by a rectified alternating current.

this case act as a motor only during the intervals when the driving torque has a positive value, and it will act as a *generator* during the intervals for which the torque is a negative or retarding one. The power curve for the motor would consist of positive and negative half-waves, resembling the full-line curve of Fig. 51.

127. Let us suppose that in the case of a loaded synchronous motor the exciting current has been so adjusted that the armature current is in step with (*i.e.*, opposite in phase to) the E.M.F. Let the exciting current now be weakened. The immediate effect of this on the motor is to diminish its E.M.F., allowing a much stronger current to flow through its armature. The motor will thus be *momentarily accelerated*, its E.M.F. getting slightly out of step with the current, so that when an armature coil is in the position shown in Fig. 86*a* there will be a current flowing through it, as indicated by the arrows. This current will, as already explained, tend to strengthen the field, thus counteracting to some extent the effect produced by weakening the exciting current. The motor will *lead*—*i.e.*, its E.M.F. will pass through its zero value before the current. That the latter must now be greater than it was when in step with the E.M.F. is also evident from the following consideration: the current not being in step with the E.M.F., the driving torque will now, during a certain fraction of each half-period, be negative; hence, in order to obtain the same driving torque, the positive torque must have a larger value than before—*i.e.*, the armature current must be greater.

The more the field excitation is reduced, the greater must the armature current become for a given torque, and the more will it be out of step with the E.M.F. A stage is finally reached when the field is so weak as to be unable, in spite of the very large armature current, to furnish the torque required. The motor then drops out of synchronism, and comes to a dead stop.

Instead of reducing the exciting current which makes the E.M.F. exactly opposite in phase to the current, let us next *increase* it. The immediate effect is to increase the E.M.F. of the motor, thus reducing the current and driving torque. The motor will therefore be *momentarily retarded.* The reversal of current in the armature coils will thus take place before the position of zero E.M.F. is reached, so that in this latter position (as shown in Fig. 86*b*) there will be a current tending to *weaken* the field, thus counteracting the effect produced by increasing the exciting current. The slight retardation of the motor armature will go on until sufficient current passes through it to provide the requisite torque; the motor will then resume its synchronous speed. The current taken by the motor must, as before, be greater than that corresponding to exactly opposite phases of current and E.M.F.; for now we have alternately positive and negative torques exerted by the motor, so that the positive impulses must be greater than before, necessitating a larger armature current.

From the above considerations it is evident that for a given load the armature current will depend on the field excitation. With a very weak field, the armature current is large, and the motor current and E.M.F. are considerably out of step. As the field is strengthened, the armature current diminishes, and at the same time comes more nearly into exact opposition of phase with regard to the E.M.F. For a certain value of the exciting current, the current in the motor armature is exactly opposed in phase to the E.M.F. This corresponds to the *minimum* R.M.S. value of the armature current, for the driving impulses received by the motor are now all positive. Any further increase of the exciting current produces an *increase* of the armature current. The relation between the exciting current and that through the motor armature is shown by the curves in Fig. 87. Corresponding to each load we have one such curve. As the load increases, the curves become more rounded.

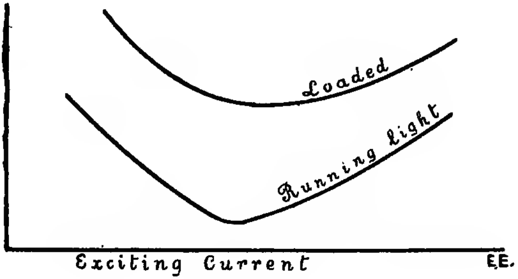

FIG. 87.

128. When the current is in exact opposition to the motor E.M.F.—*i.e.*, when it has its minimum value—it is obvious that the efficiency of working is at its maximum. For the loss due to heating of the line

and generator and motor armatures is then reduced to a minimum. In practice it is usual to excite the motor a little *above* the point corresponding to exact opposition of phase. This renders it more stable than when it is excited a little below this point, as will be evident from the following considerations.

Suppose, for the sake of simplicity, that E.M.F. and current obey the simple sine law, and that E and I stand for the maxima values of these quantities. Let γ denote the phase angle of displacement between the zero values of E.M.F. and current. Then the power spent in actually driving the motor is, by Art. 61, $\frac{1}{2}$ E I cos γ. Let now the load be suddenly increased. This will, of course, produce a momentary retardation of the motor armature. A reference to Fig. 86 will easily convince the student that the effect of such a retardation is to weaken the motor field, whether the exciting current be strong or weak, thus allowing a larger current to pass into the motor armature, and increasing the torque. But with a strong field, the value of E in the product $\frac{1}{2}$ E I cos γ is* greater than with a weak field. Hence the maximum value of the product will be greater for the stronger field, and the motor will not fall out of synchronism so easily as when the field is weak. Where the motor is subject to sudden changes of load, a slight over-excitation† is thus useful in imparting greater stability to the motor, and providing a larger margin of power.

129. Ordinary alternators when used as synchronous

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* For a given value of I cos. γ.

† By an "over-excited motor" is here meant a motor whose exciting current is greater than that corresponding to exact opposition of phase between armature E.M.F. and current.

motors possess the disadvantage of not being self-starting. For starting them, an auxiliary motor must be used. In cases where the alternator is excited by a small continuous-current dynamo, this dynamo may be used as a motor to start the alternator and bring it up to the speed of synchronism. For this purpose, however, a battery of secondary cells must be provided. In other cases special forms of small self-starting alternate-current motors may be used for bringing the large motor up to its speed of synchronism. From this it is evident that ordinary alternators cannot be regarded as satisfactory alternate-current motors for general purposes of power distribution. In some cases, however, they may be used with advantage, and where a constant speed is required they are especially suitable.

130. The problem of a synchronous motor is closely connected with that of the parallel working of alternators. Before considering this subject we shall briefly explain wherein lie the advantages of parallel working.

Let us suppose that a central station contains a certain number of generating units. The area of supply will require a number of feeding points where the current from the generating station enters the distributing mains. The feeders which supply the various districts may be arranged in a variety of ways at the generating station. One method (largely in vogue in America) is to divide the feeders into a number of groups, and allow each alternator to supply one group of feeders. If no arrangement is made for changing over from one alternator to another during the hours of light load, such a system is extremely wasteful, for large engines and alternators have to be kept running for many hours at a fraction of their full load. For this reason, arrangements are generally made at the station for transferring feeders from one alternator to another. Before such a transfer can be made, however, the circuit must be broken, giving rise to a momentary extinction of all the lamps supplied by the feeders which are being changed over. Such breaks in the continuity of the service are objectionable, and may be entirely avoided by running all the alternators in parallel. Again, supposing a sudden breakdown of one of the alternators to take place during the hours of heavy load, considerable delay would probably arise before the corresponding group of feeders could be switched into the circuit of a spare generator.

When the alternators are run in parallel, all the feeders may be connected to "'bus" bars, and the alternators which are in circuit may be kept running at or about their full load, thus working with maximum efficiency. As the load increases, more alternators are switched into circuit, the supply of power being always kept adjusted to the demand. If one of the alternators breaks down, it may be immediately cut out of circuit without in any way interrupting the continuity of the supply, the extra load being equally divided between the other alternators which are in circuit. As regards convenience, simplicity, and economy of working, parallel running has therefore very great advantages.

131. We have already seen that when two alter-

nators are synchronised and the switch connecting them is closed, they will tend to run in synchronism even if one of them is loaded. When both alternators are driven by engines, we should therefore expect them to run in synchronism without much difficulty, and this is found to be the case in practice. So powerful is the synchronising action of most alternators, that in many cases an alternator which is not in step—*i.e.*, not in direct opposition of phase—with the E.M.F. of supply, when thrown into circuit will be pulled round so as to come into opposition of phase.* The mere synchronous running of two alternators does not, therefore, present any difficulty. But *the successful parallel working of alternators is a very different matter from mere synchronous running*. It is obvious that an alternator might be kept running in parallel with others, and not only contribute nothing towards the combined output, but actually absorb power. Successful parallel running depends on the ease with which each alternator will immediately take up its share of the load as it is switched into circuit, and with which it will keep in exact step with all the others *during each period.*

132. The parallel running of alternators may be conveniently studied in two aspects—an electrical and a mechanical one. We shall consider the electrical aspect of the problem first.

The operation of throwing an alternator into circuit has already been practically described (Art. 124). The alternator is brought into step and synchronism by means of the synchronising transformer, and the exciting current is adjusted so that the E.M.F. of the alternator on open circuit is somewhat higher than that of the station. The switch is then closed, and the steam supply to the engine is increased until the alternator ammeter indicates that the machine is taking its fair share of the load.

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* Such a method of procedure is, however, not advisable (Art. 124, footnote), as it might result in serious damage to the alternator coils.

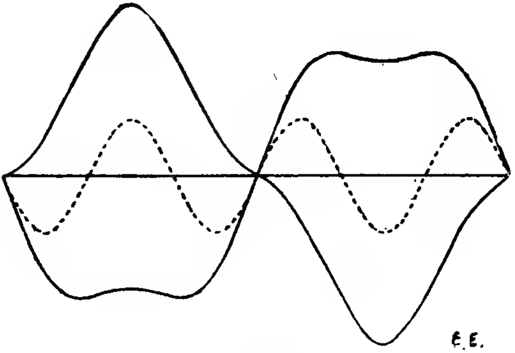

Fig. 88.

As an ideal case, suppose all the alternators to run at a perfectly constant speed, and to be in exact step with each other. If the E.M.F. waves of all the alternators are precisely alike, then it is obvious that no interchange of current will take place between any two machines. But if the E.M.F. waves of two machines are widely different, such as the peaked and flat-topped waves shown in Fig. 88, then there will be a resultant E.M.F. (shown dotted) in the local circuit, A B C D E F G H (Fig. 89), and consequently there will be a current in this circuit whose periodicity is different from that of the main current. Since this *local* current does not pass into the external circuit, it contributes nothing towards the useful output of the machines, but reduces the power factor, and lowers the efficiency on account of the additional heating of the machines with the larger current. It is thus evident that for parallel working it is desirable to use machines of the same type.*

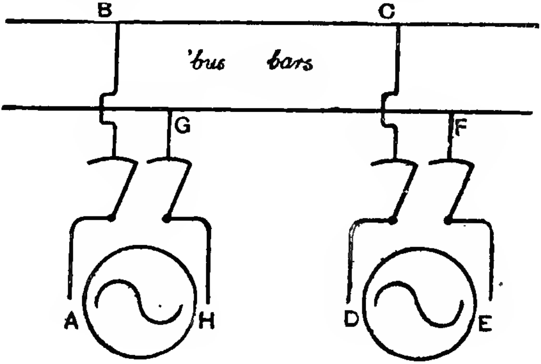

Fig. 89.

133. Perfect constancy of speed is unattainable in practice, so that we have to consider the effects produced by a slight *momentary* retardation or acceleration

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* This does not mean, of course, that machines giving widely different E.M.F. waves will not run in parallel; but merely that machines giving similar E.M.F. waves will work more economically (other things being equal) than those whose waves differ considerably. The exigencies arising in practice sometimes necessitate the parallel working of widely different types of machines.

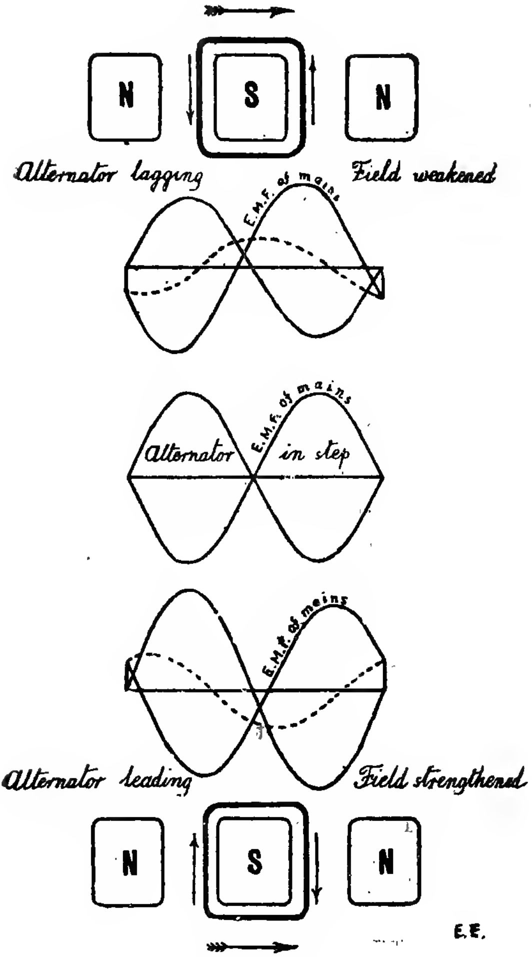
N
S
N
Alternator lagging
Field weakened
E.M.F. of mains
Alternator
E.M.F. of mains
in step
E.M.F. of mains
Alternator leading
Field strengthened
N
S
N
E.E.

Fig. 90.

of one of the alternators. Let one of the machines be momentarily retarded. In Fig. 90, the middle curves show the phase relation between the E.M.F. of the mains and that of the machine under consideration when there is exact opposition of phase.* A slight momentary retardation of the machine will produce a lag of its E.M.F. behind that of the mains, so that the balance of E.M.F.'s in the local circuit formed by the machines will be destroyed, and the resultant local E.M.F. (show dotted in Fig. 90) will produce a current which is almost in quadrature with the E.M.F. of the machine. This current will exert a powerful effect on the field of the machine, for it reaches its maximum value when the alternator coils are almost facing the poles, and are thus in the best position for exerting a magnetic effect. A reference to the upper part of the diagram (Fig. 90) will readily enable the student to see that a slight *lagging* of the alternator behind the other machines results in a *weakening* of its field. Its E.M.F. will therefore be reduced, and hence also the load current which it supplies to the external circuit. The lagging alternator is thus automatically relieved of part or the whole of its load, and in extreme cases even supplied with power from the other machines.

Suppose next that the alternator is momentarily *accelerated.* Reference to the lower part of Fig. 90 shows that in this case a local current is produced which *strengthens* the field. The E.M.F. of the machine is increased, and hence also its load. Any tendency of the machine to get ahead of the others is thus checked by an immediate increase of load.

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* In what follows, it is supposed that all the machines give simple harmonic E.M.F.'s.

These automatic adjustments are, as we have seen, produced by a local current between the machines which controls their fields. Hence this local current is frequently termed the *controlling current;* it performs the function of maintaining the machines in synchronism.

134. So far we have merely considered the parallel running of alternators with respect to the electrical actions by which the machines are maintained in synchronism. From the explanations given above it follows that unless the momentary retardations and accelerations which are liable to occur in the case of any given machine are small, there will be a large interchange of current between the machines, which will considerably reduce the power factor, and render parallel working unsatisfactory. But constancy of speed depends entirely on the prime mover. It is thus evident that the successful running of alternators in parallel is mainly dependent on the *prime movers* driving them. From this point of view, turbines and engines with three cranks (set 120deg. apart, so as to give the most even turning moment possible) are much more suitable for working alternators in parallel than single-crank steam-engines. Gas or oil engines working on the Otto cycle are totally unsuitable, for the torque exerted by such engines, even when fitted with heavy flywheels, fluctuates considerably during the engine cycle.

For successful parallel running, the engines driving the alternators should not be provided with independent governors. Otherwise, unless the governors are adjusted with a precision which is impossible in practice, there will be difficulty in getting each machine to take its fair share of the load, the machines which tend to run faster taking too much load, and those which tend to run slower too little. The supply of steam should be adjusted by hand until the machine is properly loaded, the governors only coming into play when owing to some accident—such as the blowing of a fuse and consequent sudden removal of the load—the engine tends to race. The governor should thus be used merely as a safety device to prevent the engine speed from rising above a safe limit, and the actual regulation of the steam supply should be done by hand.

When the problem of the parallel running of alternators first presented itself, it was supposed that whether two machines would or would not run in parallel was a question dependent entirely on the kind of alternator used (*e.g.*, on whether the alternator armature did or did not contain iron, etc.). The above explanations will, however, make clear to the student that although the controlling current (Art. 133) which passes between any two machines is to some extent determined by the types of the machines, yet the main factor determining it is not the type of alternator, but the *type of engine.* Successful parallel running is determined almost entirely by the capability of the prime mover to maintain a constant torque.

Notes on Chapter XV.

Dr. Hopkinson was the first to give (in 1884) a theory of the parallel running of alternators, and the behaviour of alternators when used as motors. Previous to Dr. Hopkinson's theoretical investigations, however, Mr. Wilde had shown experimentally that alternators could be made to run in parallel. Dr. Hopkinson's main conclusions were put to the test of experiment at the South Foreland lighthouse, and shown to be substantially correct.

The most important subsequent contributions to this subject were made by Mr. Mordey, who was led to challenge some of Dr. Hopkinson's conclusions by certain results obtained with the then newly constructed Victoria-Mordey alternators; by Mr. Kapp; and by Mr. Swinburne, who advanced the theory sketched out in the present chapter, and which is known as the "armature reaction" theory, as opposed to the older theory advanced by Dr. Hopkinson, which may be termed the "self-induction" theory. The difference between these two theories may be briefly stated thus: According to the "self-induction" theory, when the alternator is running under load the pulsations of the armature current produce, on account of the inductance of the armature, a pulsating magnetic field which is superposed on the original field of the alternator due to its exciting current. The resultant field is, according to this theory, one which fluctuates within wide limits. If the theory be true, then fluctuations of the field will give rise to an alternating E.M.F. in the exciting coils (the armature and field winding behaving like the primary and secondary of a transformer respectively), so that the exciting current will undergo fluctuations. This effect would undoubtedly take place in an alternator with a perfectly laminated field. But with fields which are either not at all or only imperfectly laminated, no such fluctuation has been detected. The conclusion, then, is that *no pulsation of the field which is practically appreciable takes place.* This conclusion is precisely the starting-point of Mr. Swinburne's "armature reaction" theory, according to which the field may be distorted and strengthened or weakened by the armature current, but cannot be made to undergo fluctuations, the latter being practically prevented by the action of eddy currents in the field cores.

CHAPTER XVI.

135. No account of the principles which underlie alternate-current working would be complete without a reference to *polyphase* alternating currents, which are coming largely into use for purposes of power distribution. Polyphase currents enable us to produce a *rotating magnetic field* without any rotation of the mass of iron which carries the exciting coils. Such a rotating magnetic field may be used for constructing motors which are self-starting, non-synchronous (Art. 123), and which require no commutator or sliding contacts of any description.

136. Let us imagine the field magnets of an ordinary continuous-current dynamo mounted so as to be capable of rotating about the armature axis; let the armature be an ordinary ring or drum armature, but let the windings, instead of being connected to a commutator, form closed independent circuits (Fig. 91) surrounding the core, each coil being short-circuited by having its two ends soldered together; or we may suppose that the armature is not in any way interfered with, but that a tight-fitting ring of copper is slipped over the commutator so as to short-circuit the coils. Let the brushes be removed, and the field winding disconnected entirely from the armature and terminals of the machine, and connected to two contact rings capable of revolving with the magnets.

Fig. 91 shows the arrangement diagrammatically. Both field magnets and armature are now capable of rotation. Let the field magnets be excited by means of a secondary battery connected to two brushes pressing against the contact rings, and let the magnets be driven round right-handedly, as shown in Fig. 91. Currents will be induced in the short-circuited coils by the revolving field, the direction of these currents being the same as that which would result if the field were kept stationary and the armature driven round in the opposite

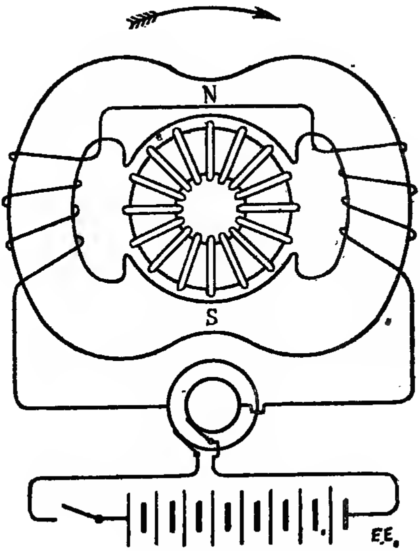


FIG. 91.

direction. The currents induced in the armature coils tend to resist relative motion of the field and armature; hence the latter experiences a torque, and begins to run in the same direction as the field. The torque acting on the armature is proportional to the product of the field intensity and the current in the short-circuited coils. If the armature is not loaded, a very small current will be sufficient to provide the torque required to overcome the frictional resistances to its motion. But if a brake is applied to the armature pulley a much larger torque must be provided, and hence, assuming the field to remain constant, a larger current in the short-circuited coils. Now, the current induced in the coils is proportional to the E.M.F., and the latter depends on the *relative* velocity of field and armature, generally called the *slip*. The E.M.F. is directly proportional to the slip. Hence, as the load is increased, the slip increases (up to a certain limit) in direct proportion to the load —*i.e.*, the speed of the armature diminishes as the load is increased.

137. We may imagine the armature rigidly coupled to a line of shafting driving machinery, and we may suppose that the engine, instead of driving the shafting directly, is made to drive the field magnet of the dynamo. The torque communicated to the field magnet by the engine will then be transmitted to the armature, which will tend to follow the motion of the magnet. Our modified dynamo, in fact, forms a sort of flexible electromagnetic coupling between the engine and the line of shafting. If the engine be supposed to run at a perfectly constant speed, the

speed of the shafting will not be constant, but will depend on the load, since the armature slip, as we have seen, increases with an increase of load.

The resistance of the short-circuited coils being extremely low, a very small induced E.M.F. will be sufficient to give rise to a powerful current. Hence the slip is never very great. The slip is conveniently expressed as a percentage of the field velocity. We may also express it in revolutions per second or per minute.

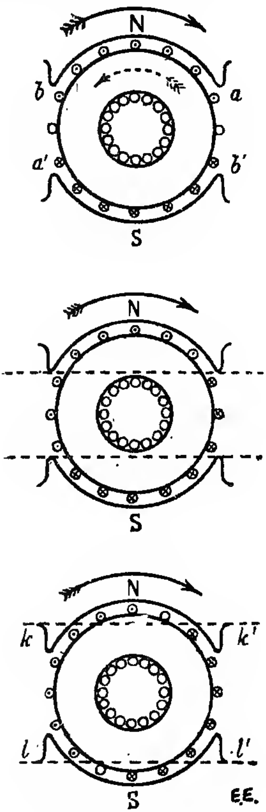
It is obvious that (as in the case of a dynamo) the current in each coil undergoes a double reversal during each *slip* revolution. The induced E.M.F. begins to act on each conductor as it enters the fringe (*a*, Fig. 92) of the magnetic field under the polar tip, and continues to act on it so long as it is under the polar surface; the E.M.F. then rapidly diminishes as the conductor passes out of the field through the fringe, *b*, on the other side, and during the motion of the conductor across the interpolar space there is no impressed E.M.F. acting on it. When it reaches the fringe, *a′*, of the opposite polar surface another E.M.F., acting in the reverse direction, makes its appearance, and continues to act until the conductor once more passes out of the field at *b′*.

The directions of the E.M.F.'s impressed on the various conductors are shown in Fig. 92, the field being supposed to rotate clockwise. The motion of the armature relatively to the field will then be counter-clockwise, as shown by the dotted arrow. The conductors marked with dots have E.M.F.'s impressed on them which act from back to front, and those marked with crosses E.M.F.'s which act from front to back of the armature. The conductors out of the magnetic field are obviously not subjected to any impressed E.M.F.'s.

138. The variations of current which take place in each coil during one slip revolution depend not merely on the impressed E.M.F. and resistance, but also on the *inductance* of each coil (Arts. 33-36). The rapidity with which a current establishes itself in a circuit depends on the time-constant (Art. 35) of the circuit. If the time-constant of the short-circuited coil were negligible, the changes of current in it would follow precisely the changes of impressed E.M.F., and for any given relative position of the field and armature, the currents would be distributed as in Fig. 92. For such a distribution of currents, we notice that the *force acting on each conductor tends to accelerate the rotation of the armature;* all the conductors in the upper half of Fig. 92 tending to move across the field from left to right, while all those in the lower half tend to move from right to left.

139. The time-constant, however, of the short-circuited coils is by no means negligible: their resistance is relatively low, and their inductance, owing to the presence of the iron core, is high. The effect produced by the retardation of both the rise and decay of current in a short-circuited coil will to a large extent depend on the slip of the armature. If the slip is very small, then the armature (although rotating rapidly in space) revolves comparatively slowly with regard to the field. As a result, the current in a conductor will have time to decay almost completely

during the motion over the neutral or interpolar space, and it will have time to establish itself soon after the



Figs. 92, 93, and 94.

application of the impressed E.M.F. But if the slip be considerable, then, owing to the more rapid rotation of the armature relatively to the field, the current in the short-circuited coil may persist during the whole of the movement of the conductor over the neutral space, and the conductor may move over an appreciable portion of the field before a current in the opposite direction is fully established in it. The distribution of curents in such a case is shown in Fig. 93.

If the slip is very great, and the time-constant of the windings considerable, then it may happen that the current persists in a conductor long after the latter has entered the reversing field. We may then have a distribution of currents such as that shown in Fig. 94. One important result is at once apparent: under each pole-piece we now have a number of conductors, some of which tend to move in one direction, others in the opposite direction. The actual torque of the armature will be determined by the differential action between the two groups of conductors under each pole-piece. The torque will be greatly reduced, both on account of this differential action and the weakening of the field due to the demagnetising belt of conductors included between the planes $k\ k'$ and $l\ l'$ (Fig. 94.)

140. The above considerations show that it would be undesirable to have armature windings with a large time-constant. The effects considered in the last article are most marked at starting. Suppose the armature to be at rest, and the field magnet rotating at its normal speed, but without being excited. Let now the exciting current be suddenly switched on. The armature slip at starting is

enormous, being equal to the speed of rotation of the magnet. Hence the effects we have just considered will be very marked, and the starting torque of the armature small, in spite of the fact that the currents circulating in the armature coils may be abnormally large. As the armature gains speed, however (supposing the initial torque to be large enough to start it against its load), the changes of current become more gradual, and although the actual currents in the coils are reduced, the differential action and demagnetising effect are also reduced, and the torque increases. This increase of torque will go on up to a certain limit. Beyond that, any further increase of speed will so much reduce the slip that the currents in the conductors will be materially weakened, and the torque will diminish.* Hence

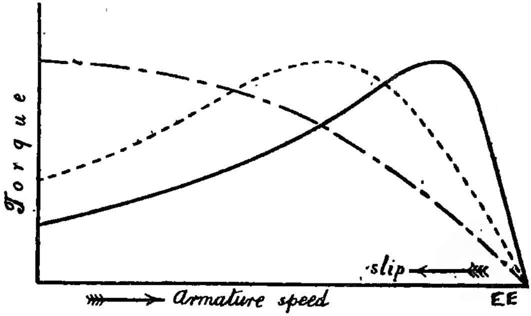


FIG. 95.

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* Beyond the point of maximum torque, the *slip* will practically vary directly as the load (Art. 136). This is another way of stating the fact that the curves of Fig. 95 are approximately straight lines for small values of the slip.

the curve connecting torque with speed or slip will be of the shape shown in Fig. 95 (full-line curve).

Suppose we have the means of varying the resistance of the armature coils (without altering their inductance), then by an increase of their resistance we can reduce the time-constant and so obtain the maximum torque with a greater slip, as shown by the dotted curve of Fig. 95. By increasing the resistance still further, we can even cause the torque to be greatest at starting, as shown by the chain-dotted curve of Fig. 95. It will thus be evident that by providing means for introducing non-inductive resistances into the armature coils, we can obtain complete control over the torque of the motor.

141. We have now investigated the behaviour of our electromagnetic coupling, consisting of an armature with short-circuited coils and a field magnet revolving round it. The only function, however, which the revolving field magnet performs is that of providing *a rotating magnetic field.* Now, a rotating magnetic field may be produced in a *stationary* mass of iron by two or more alternating currents which differ in phase. Let the revolving magnet be replaced by a *stationary* one, so wound that when suitable alternating currents are sent through its coils it gives rise to a rotating magnetic field. Then we have transformed our electromagnetic coupling into a *motor.* Such motors are variously termed *polyphase* motors, *rotary-current* motors, or *induction* motors.

Notes on Chapter XVI.

The first mention of a rotary-field motor occurs in a paper by Mr. Walter Baily, published in the *Philosophical Magazine* for

October, 1879. A year later—in 1880—M. Marcel Deprez constructed what was practically a two-phase rotary-field motor, the currents for driving it being obtained from a battery by means of two two-part commutators. In 1885, Prof. Galileo Ferraris carried out a number of experiments with rotating magnetic fields, and an account of these, along with a simple theory, was published in 1888. To Mr. Nikola Tesla belongs the credit of having produced, in the same year, rotary-field motors which could be regarded as articles of commerce.

EXERCISES.

1. A rectangular frame of wire, 18in. × 10in., is rotated in a magnetic field of intensity = 8,000 with a speed of 500 revolutions per minute, the field itself rotating at 600 revolutions per minute. What is the maximum value of the E.M.F. induced in the frame?

2. The resistance of a short-circuited coil is 2 ohms, and its inductance ·02 henry. When the coil is rotated in a stationary uniform field at a speed of 800 revolutions per minute, a current of 5 amperes (R.M.S. value) is found to flow in it. If the field itself be made to rotate in the same direction at 900 revolutions per minute, what will be the value of the current in the coil, and of the mean torque exerted on the coil by the rotating field?

CHAPTER XVII.

142. In the present chapter we shall consider how a rotating magnetic field may be produced by means of two or more alternating currents which differ in phase. Such combinations of alternating currents are termed *polyphase* or *multiphase* currents. Lower down (Art. 149) we shall explain how polyphase currents may be generated.

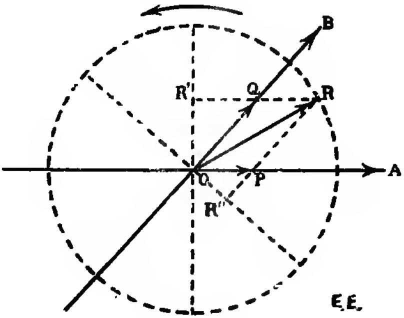

Fig. 96.

143. Any vector may, as we have seen, be replaced by two component vectors (Art. 11) lying along any two given straight lines. Thus we found that the single vector O C in Fig. 8 was equivalent to the two vectors O A and O B. Now a rotating magnetic field is conveniently represented by a rotating vector. Hence, if we take the rotating vector O R (Fig. 96)

to represent a rotating magnetic field, we may at any given instant replace it by two component vectors O P and O Q, lying along two fixed straight lines. Let the angle between the positive directions of the two lines be a; let the initial position of the rotating vector be along one of the fixed straight lines, O A, and let its angular velocity be p. After t seconds, the rotating vector will occupy the position O R, where A O R $= p\,t$. Since R Q is always parallel to A O, it follows that R Q produced will be perpendicular to O R′, a line drawn through O at right angles to O A. Hence O R′ will be the projection of both O R and O Q, and we shall have (Art. 12)

$$OR' = OR \cos R'OR = OQ \cos R'OQ,$$

i.e., since $\cos R'OR = \sin p\,t$, and $\cos R'OQ = \sin a$,

$$OR \sin p\,t = OQ \sin a,$$

whence
$$OQ = \frac{OR}{\sin a} \sin p\,t.$$

Now, since a is constant, it follows that O Q is an alternating vector of the same periodicity as the rotating vector, O R, and of amplitude $= \dfrac{OR}{\sin a}$.

Similarly, by considering the projection O R″ of O R or O P on a line through O drawn at right angles to O B, we find that

$$OP \sin a = OR \sin (a - p\,t),$$

or,

$$OP = -\frac{OR}{\sin a} \sin (p\,t - a), \text{ by Art. 8, equation (4),}$$

$$= \frac{O\,R}{\sin \alpha} \sin (p\,t + \pi - \alpha),$$

since $\sin \theta = - \sin (\theta + \pi)$.

We have thus shown that a *rotating* magnetic field will be obtained if two *alternating* magnetic fields of equal amplitude and the same periodicity be superposed on each other, provided there is a phase difference between the alternating fields equal to the supplement of the angle which their positive directions make with each other. The intensity of the resultant rotating field is equal to the amplitude of either of its components, multiplied by the sine of the angle between them; and the period of the resultant field is the same as that of its components.

144. The above results at once suggest a method of producing a rotating magnetic field by means of two alternating currents differing in phase. For if we take a coil of wire and send an alternating current through it, we obtain an alternating magnetic field. If we use two coils set at a certain angle to each other, as shown in plan and elevation in Fig. 97, and send two alternating currents having the proper phase difference through the coils, the magnetic field at the common centre of the two coils will be a rotating one, the speed of rotation being equal to the periodicity of the alternating currents. Supposing α to denote the angle between the positive directions of the magnetic fields due to the coils when continuous currents which are reckoned positive are sent through them, it follows by Art. 143 that the alternating currents used for

producing a rotating field inside the coils must differ in phase by $\pi - \alpha$.

In most cases the phase difference between the alternating currents is already determined, and cannot be varied; we may then adjust the angular position of the two coils so as to produce a rotating field.

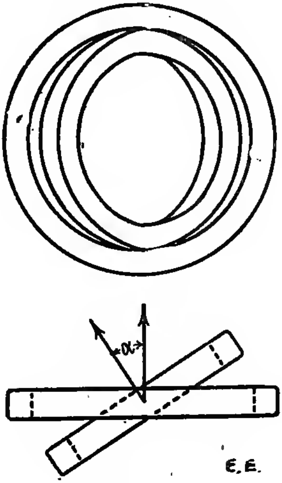

FIG. 97.

145. In practice, when two alternating currents differing in phase (or *two-phase* currents) are used, the phase difference between them is generally made equal to $\frac{\pi}{2}$. Hence the angles which the two coils producing the rotating field must make with each other is also equal to $\frac{\pi}{2}$—*i.e.*, the coils must be at right angles.

to each other. In producing a rotating field for an induction motor (Art. 141), it is an obvious advantage to provide the exciting coils with laminated iron cores, for the magnetic reluctance of the circuit is thereby very materially reduced, and the magnetising current required is diminished. The field magnet of such a motor frequently takes the shape of a hollow laminated iron cylinder surrounding the armature, the two sets of coils being arranged at the extremities of two diameters at right angles to each other, as shown diagrammatically in Fig. 98. If a continuous current having the direction indicated in the figure be sent through the pair of coils marked I., a horizontal magnetic field is produced inside the cylinder; a continuous current through the pair of coils II. gives rise to a vertical field. Two currents in quadrature with each other sent through the two

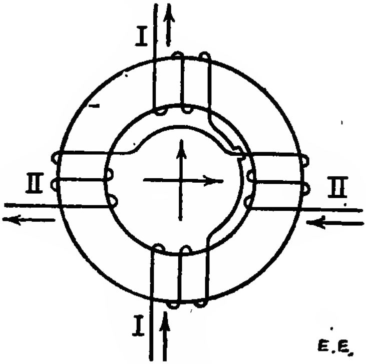

FIG. 98.

sets of coils will give rise to a rotating magnetic field, as explained above.

146. A rotating magnetic field, as we have just shown, may be produced by superposing two alternating fields of equal amplitude. Conversely, an alternating field may be produced by the superposition of two rotating fields of equal intensity if they are made to rotate with equal speeds in opposite directions. Let the two rotating fields be represented by two rotating vectors. There will obviously be instants at which the vectors become coincident. Let the line O A (Fig. 99) denote the position of the rotating vectors when these are coincident. The resultant vector for the position of coincidence is obviously the sum of the two vectors—*i.e.*, it is a vector along O A of double the length. After a time

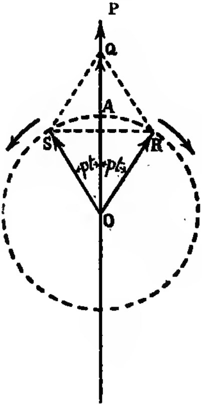

Fig. 99.

t, each rotating vector will have traced out an angle $p\,t$ (where p is the angular velocity) so that the two vectors are equally inclined to the line O A. Resolving each vector into a horizontal and a vertical component, we have, if R denote the length of each rotating vector,

sum of horizontal components $= \mathrm{R} \sin p\,t - \mathrm{R} \sin p\,t = 0$;

sum of vertical components $= \mathrm{R} \cos p\,t + \mathrm{R} \cos p\,t =$

$2\,\mathrm{R} \cos p\,t$.

Hence the resultant vector is always vertical, and we see that it is an alternating vector of amplitude 2 R, and of period equal to that of the rotating vectors.

If we have an alternating vector of amplitude O P (Fig. 99), and if at a given instant its value is O Q, then in order to find the corresponding positions O R and O S of the two rotating vectors into which it may be resolved, it is obviously only necessary to bisect O Q, through the point of bi-section to draw a perpendicular to O P, and with O as centre and radius equal to $\frac{1}{2}$ O P to describe arcs cutting the perpendicular in R and S. Then O R and O S give the instantaneous positions of the two rotating vectors.

147. Instead of the arrangement shown in Fig. 98, in which there are two sets of coils supplied with two currents in quadrature with each other, let us next consider an arrangement of three pairs of coils and three currents, as shown in Fig. 100. If the positive directions of the currents be those indicated by the arrows, then the corresponding magnetic fields will have the directions O A, O B, and O C as indicated in

the figure. Let the currents flowing through the three pairs of coils be denoted by I., II., and III., and let us suppose that they are alternating currents of equal magnitude but differing in phase by $\frac{2\pi}{3}$ (or 120deg.). Then the numerical values of the three magnetic fields due to the three currents may be approximately represented by the lengths of the projections on the vertical axis of three equal rotating vectors, 1, 2, and 3, as shown in Fig. 101, *a*.* Now, each of the three alternating magnetic fields may be replaced in accordance with the results deduced in Art. 146 by two equivalent rotating fields. In order to find the relative positions of the six rotating vectors which are equivalent to the three alternating vectors O A, O B, and O C (of Fig. 100), we may choose the instant corresponding to Fig. 101, *a*, at which the vector O A

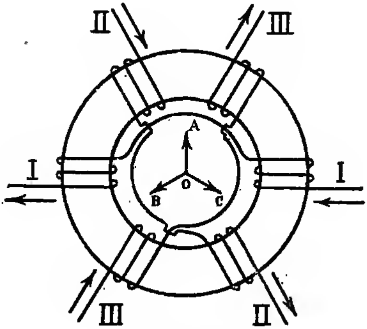

Fig. 100.

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* The permeability being supposed constant.

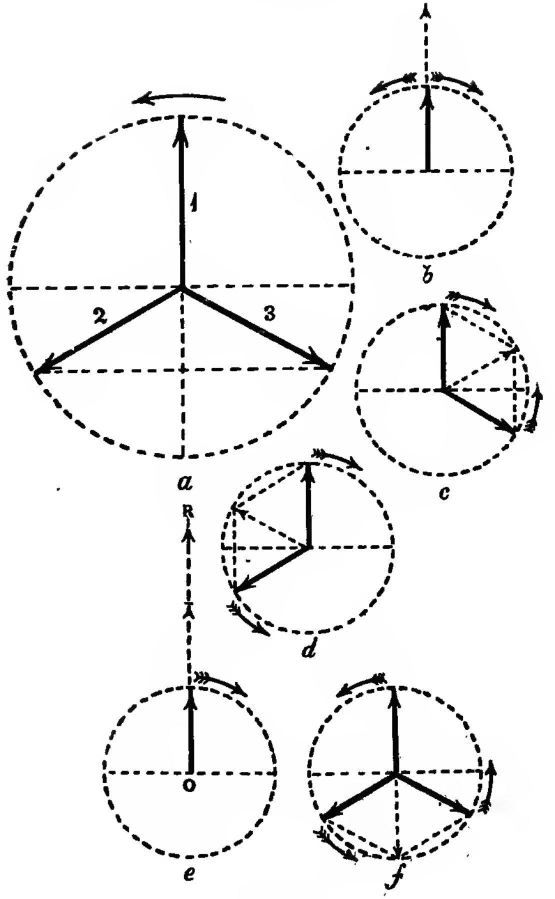

FIG. 101.

has reached its maximum positive value, while O B and O C are both negative, the former increasing numerically and the latter diminishing. Applying the result deduced in the latter part of Art. 146, we find that at the instant considered the positions of the rotating vectors which are equivalent to the alternating vectors O A, O B, and O C, are as shown in Fig. 101, *b*, *c*, and *d* respectively. Taking the three component vectors which rotate right-handedly and compounding them, we obtain a single rotating vector O R (Fig. 101, *e*). Again, taking the left-handedly rotating vectors, we find that they balance each other at every instant, as is evident from Fig. 101, *f*. We thus see that the resultant of the six rotating vectors is a single rotating vector. In other words, the three alternating magnetic fields O A, O B, and O C (Fig. 100), which differ $\frac{2\pi}{3}$ in phase, yield when compounded a simple rotating field. An inspection of the diagrams will easily enable the student to see that the intensity of the resultant revolving field is equal to 1·5 times the amplitude of one of the component alternating fields.

148. We have thus shown that a rotating magnetic field may be produced by using three alternating currents which differ $\frac{2\pi}{3}$ in phase. Such a system of currents is known as a *three-phase* or *Drehstrom* system. It may be shown that a rotating field may be produced by the use of four, five, or a larger number of alternating currents with suitable phase differences between them; but since the polyphase systems used in practice are all either two-phase or three-phase systems, we shall restrict our attention to these last two.

Notes on Chapter XVII.

In practice, the method of producing a rotating magnetic field by the superposition of *two* simple alternating fields at right angles to each other and having a phase difference of $\frac{\pi}{2}$, was first used in America by Mr. Tesla.

In the Langdon-Davies motor, a rotating field is obtained by superposing two alternating fields the angle between which is greater, and the phase difference less, than $\frac{\pi}{2}$, the sum of these two quantities being approximately $= \pi$.

On the Continent, the earliest types of rotary-field motors were supplied with *three*-phase currents, the rotating field being obtained by the superposition of three alternating fields with phase differences of $\frac{2\pi}{3}$. Herr Dolivo Dobrowolsky, who was the first to introduce such three-phase or *Drehstrom* motors, claimed for them a superiority over the Tesla or two-phase ones, on the ground that a more even driving torque was obtained with the former. Experimental investigation has, however, shown that this advantage is non-existent.

Exercises.

1. If the phase difference between two alternating magnetic fields be $\frac{\pi}{4}$ (45deg.), what angle must their directions make with each other in order that the resultant field may be a uniformly rotating one of constant intensity?

2. Two alternating magnetic fields of the same periodicity and of amplitudes 3,000 and 4,000 respectively, are produced along two straight lines at right angles to each other, the phase difference between the fields being $\frac{\pi}{2}$. Draw a vector diagram showing the direction and magnitude of the resultant field at intervals corresponding to $\frac{1}{20}$th of the period.

3. Two alternating magnetic fields, each of amplitude 5,000, and having a phase difference of $\frac{\pi}{4}$, act along two lines at right angles to each other. Draw a diagram to show the consecutive positions and magnitudes of the resultant field at intervals of $\frac{1}{20}$th of a period.

CHAPTER XVIII.

149. Polyphase currents may be produced by several methods. That commonly used in practice consists in providing an alternator armature with as many independent windings as there are phases, and arranging these windings in such relative positions that the required phase difference is obtained. In the case of a two-phase generator, for instance, the two armature circuits would be so placed with regard to the poles that the E.M.F. in one circuit reaches its maximum value while that in the other is passing through its zero value. This will be understood by reference to Fig. 102, which shows three poles, with a number of conductors so placed that the E.M.F. in alternate conductors is at its maximum value, while that in the intermediate conductors is zero. By connecting alternate conductors, as shown in the sketch, to form

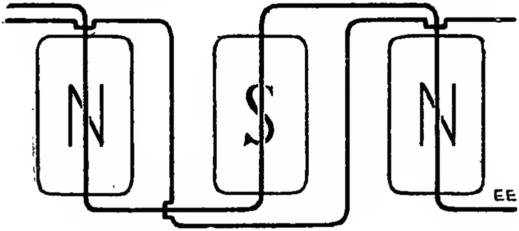

FIG. 102.

one circuit, and the intermediate conductors another, we obtain two independent circuits, the E.M.F.'s in which are of the same periodicity, but in quadrature with each other. By adopting this method of winding, we can construct an alternator which will produce two-phase currents, or a *two-phaser*.

A similar procedure may be adopted in constructing *three-phasers*, or alternators capable of yielding three currents, the phase difference between which is $\frac{2\pi}{3}$ (or 120deg.) In order to fix the relative positions of the three circuits, we divide the distance A B (Fig. 103), which corresponds to a complete period, into three equal parts. If, now, conductors be placed in the positions A, C, and D, and maintained in these relative positions on the armature, it is evident that the phase difference between the E.M.F.'s corresponding to any two of these conductors is $\frac{2\pi}{3}$, since they are placed

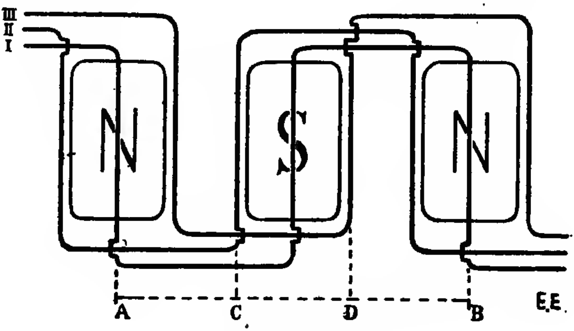

FIG. 103.

at distances apart which correspond to one-third of a period.

150. We have next to consider how two-phase and three-phase currents may be transmitted from the generating station to the area of supply. It is obvious that this may be accomplished by keeping all the circuits independent, and using two conductors for each circuit. This plan is sometimes adopted with two-phase circuits, and in that case four conductors will be required for transmitting the power. For a three-phase system this method would involve the use of six conductors. In order to avoid needless complication, reduce the number of conductors, and effect considerable economy of copper, the circuits of a three-phase system are coupled together so that the number of conductors is reduced to *three*. The circuits of a two-phase system may also be connected together, so as to require only three wires for transmitting the power.

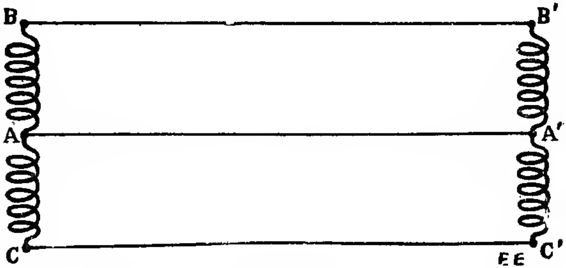

FIG. 104.

151. In Fig. 104, let A B and A C denote the two

windings of a two-phaser, and A′ B′ and A′ C′ the two windings of a two-phase motor. Instead of using four wires (two for connecting each pair of circuits), we may join the ends of the circuits to common points A and A′ and use only three wires as shown. If we assume the currents in A B and A C to follow the simple harmonic law, then the instantaneous values of these two currents may be represented by the projections on the vertical axis of two equal rotating vectors at right angles to each other, as shown in Fig. 105. Now the current which flows along A A′ (Fig. 104) is at every instant equal to the algebraical sum of the currents in B B′ and C C′ respectively; hence it may be represented by the projection on the

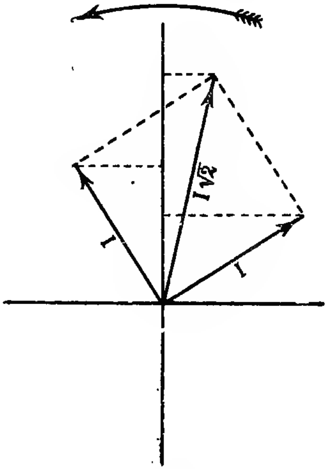

FIG. 105.

vertical axis of the diagonal of the square shown in Fig. 105. It is thus evident that the wire A A′ connecting the junctions of the two circuits carries a larger current than either of the outside wires B B′ or C C′, the ratio of the currents being $\sqrt{2} : 1$, or $1{\cdot}41 : 1$ approximately. *If we suppose that the current density is the same in all the wires*, then it is easy to calculate the saving of copper effected by the use of three wires instead of four. For if a denote the sectional area of each wire in the latter case, the total amount of copper required will be proportional to $4\,a$. With three wires, on the other hand, the amount of copper will be proportional to $2\,a + 1{\cdot}41\,a = 3{\cdot}41\,a$. We therefore find that with three wires only about 85 per cent. of the former amount of copper will be required.

The reduction in the amount of copper is, however, balanced by the better insulation of the line which must be provided with the three-wire arrangement. When four wires are used, the maximum P.D. between any two wires never exceeds the E.M.F. developed by one alternator winding. But when the windings are coupled as in Fig. 104, it is evident that the P.D. between B B′ and C C′ is much higher than the E.M.F. of one of the circuits, and for a simple sine wave is about 40 per cent. higher. The extra cost of the insulation with three wires might therefore in some cases more than balance the saving in copper, and it might be more economical to used four wires than three.

152. With a three-phase system the connections may be made as in Fig. 106. The three circuits are

connected so as to form a closed mesh, A B C, round which there is no resultant E.M.F., the E.M.F.'s in any two circuits being at every instant balanced by the E.M.F. in the third circuit. The total current which enters the mesh must at every instant be equal to the total current which leaves it. Hence if i_1, i_2, and i_3 stand for the instantaneous currents in A A′, B B′, and C C′ respectively, reckoned positive when flowing from left to right, we must have

$$i_1 + i_2 + i_3 = 0;$$

i.e., the algebraical sum of the three currents must at every instant be equal to nothing. Similarly, if i_1', i_2', and i_3' denote the instantaneous currents in A B, B C, and C A respectively, reckoned positive when flowing in a counter-clockwise direction round the mesh, we must have

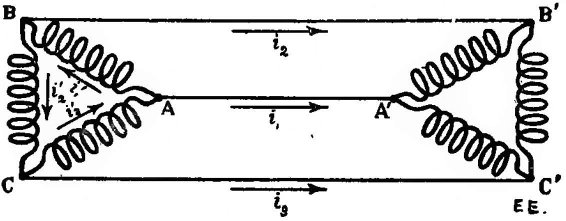

FIG. 106.

$$i_1' + i_2' + i_3' = 0,$$

since there can be no local current round the mesh (the E.M.F.'s balancing each other at every instant).

We also have the equations

$$i_1 = i_3' - i_1'; \qquad i_2 = i_1' - i_2'; \qquad i_3 = i_2' - i_3',$$

since the currents flowing towards any point must equal those flowing away from it.

If A′ B′ C′ represent the three sets of coils of an induction motor, then from the symmetry of the arrangement it follows that the currents in the three line wires will all be equal to one another. In order to find their values, we make use of the last set of equations. The currents i_1', i_2', i_3', differing 120deg. in phase, may be represented by the projections on the vertical axis of three equal rotating vectors, I_1', I_2', and I_3' inclined 120deg. to each other, as shown in Fig. 107. Using the equation

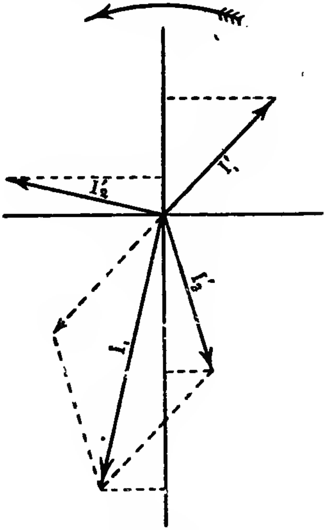

FIG. 107.

$$i_1 = i_3' - i_1',$$

we see that the current i_1 may be represented by the projection on the vertical axis of a rotating vector I_1, which is the resultant of I_3' and of I_1' reversed (the latter is shown dotted in Fig. 107). From the geometry of the figure it will be evident that

$$I_1 = 2\, I_1' \cos 30\text{deg.} = I_1' \sqrt{3} = 1{\cdot}73\, I_1'.$$

In other words, the currents in the line wires are about 73 per cent. greater than those in the alternator and motor coils.

The above method of connecting the circuits of a three-phase system is known as the *mesh* or *triangle* method. It is easy to compare the relative amounts of copper required when this method is used and when the three circuits are independent of each other and connected by means of six wires. Let the sectional area of copper in each conductor corresponding to the latter case be a. The amount of copper will then be proportional to 6 a. In the former case, when three wires are used with a *mesh* arrangement, the amount of copper is proportional to $3 \times 1{\cdot}73\, a, = 5{\cdot}19a$, so that a saving of copper amounting to over 13 per cent. is effected by using the mesh arrangement. This saving is not counterbalanced by any increased difficulties as regards insulation, for it will be seen that the P.D. between any two wires is never higher than that which would exist if the circuits were maintained independent and six wires were used.

153. Besides the *mesh* arrangement of the circuits of a three-phase system, there is another one, known as the *star* arrangement. This is shown in Fig. 108. The three circuits have one point, O (called the neutral

point), in common, and their other ends are connected to the three line wires. If i_1, i_2, and i_3 denote the instantaneous currents in O A, O B, and O C, reckoned positive when flowing away from O, then we have

$$i_1 + i_2 + i_3 = 0.$$

These three currents might, as in Fig. 107, be represented by the projections of three rotating vectors making angles of 120deg. with each other.

Let the instantaneous P.D.'s between O and A, O and B, and O and C respectively, reckoned positive when acting away from O, be noted by v_1, v_2, and v_3. We may suppose that the projections of the rotating vectors of Fig. 107 represent to a suitable scale these P.D.'s. Then, since the P.D. between A and C is equal to $v_3 - v_1$, it follows that the vector marked I_1, in Fig. 107, will represent the maximum value of the P.D. between A and C; and this, as we have seen, is 1·73 times (Art. 152) the maximum value of the P.D.'s across O A, or O B, or O C. Hence with the star arrangement the P.D. between any pair of line

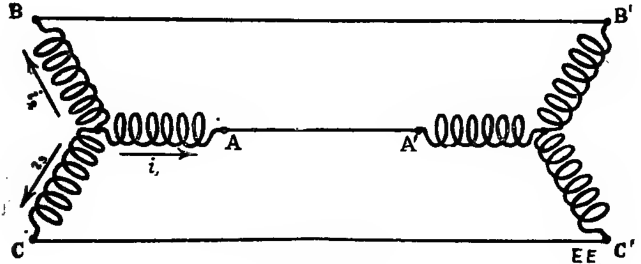

FIG. 108.

wires is over 70 per cent. higher than the P.D. across each alternator circuit.

The currents in the line wires being the same as those in the three-alternator circuits, it is obvious that the *star* arrangement results in a saving of copper, amounting to fully 50 per cent. as compared with a six-wire line.* Again, comparing the *mesh* with the *star* arrangement, we find the relative amounts of copper to be as 519 : 300, so that if we replace a *mesh* arrangement by a *star* one, we save over 42 per cent. of the copper. Against this, however, must be put the cost of the higher insulation required with the star grouping, so that in some cases the mesh arrangement might be preferable to the star.

Notes on Chapter XVIII.

The design of a polyphase generator proceeds on precisely the same lines as that of an ordinary single-phase alternator. In most modern machines, the armature coils are stationary, and the field magnet is made to rotate. By this means the insulation of the coils is not subjected to the strains which are liable to occur in a rotating armature, and the further advantage is secured that the revolving magnet (or inductor) provides a very efficient flywheel.

There is one respect in which a polyphaser differs markedly from a single-phaser. In the latter, the torque which resists the motion fluctuates between very wide limits during each period. In the former, the resisting torque is practically constant. The same remark applies to polyphase as compared with single-phase motors.

Exercises.

1. Assuming the same P.D. at the receiving station and the same loss in the line, compare the amounts of copper required for transmitting a given amount of power by means of (*a*) a single-phase current; (*b*) two-phase currents with a common return.

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* All these comparisons, be it noted, are made on the basis of a constant-current density.

CORRECTION.

The correct answers to Exercise 2 *on page* 257 *are:* (*a*) £937; (*b*) *and* (*c*) £933.

The statement on page 256 regarding the saving of copper effected by replacing a mesh arrangement by a star one is apt to be misleading. It is correct if we suppose the alternator *already wound*, and that we are at liberty to couple up its coils either mesh or star fashion—in other words, that the line P.D. is at our disposal. If, however, the line P.D. is *given* (as in Exercise 2), then we may use an alternator whose circuits are coupled either mesh or star fashion; in the former case, the coils must be so wound that each circuit supplies a P.D. equal to that between the line wires; in the latter, each circuit need only supply $\frac{1}{\sqrt{3}}$ times the line P.D. *The amount of copper is determined solely by the P.D. between the line wires, and not by the fact of the alternator coils being coupled mesh or star fashion.*†

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† I am indebted to Mr. R. W. Weekes for drawing my attention to the misleading nature of the statement on page 256.

2. It is required to transmit 500 h.p. along a line five miles long, and the loss in the line is not to exceed 10 per cent. of the power transmitted (so that the power delivered at the far end of the line is 450 h.p.). The P.D. between each pair of mains at the generating station is to be 3,000 volts. With a single-phase system of transmission, synchronous motors having a power factor of ·98 are available. For a three-phase system, the power factor of the motors would only be ·85. Taking the cost of copper at £65 per ton, find the cost of the line for the following cases: (*a*) single-phase transmission; (*b*) three-phase transmission, triangle grouping; (*c*) three-phase transmission, star grouping. Assume that the resistance of 1 yard of copper 1 square inch in section is ·0000259 ohm.

3. Assuming the same loss in the line, and the same *maximum* value of the P.D. between any two conductors, compare the relative amounts of copper required for a two-phase system of transmission with (*a*) four wires; (*b*) three wires. The *amplitude* factor* is assumed to be the same in both cases.

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* By the *amplitude factor* is meant the ratio of the R.M.S. to the maximum value.

CHAPTER XIX.

154. In order to reduce the magnetising current of induction motors, and render the magnetic leakage as small as possible, the air-gap of such motors is reduced to a mere mechanical clearance space. This is attained by embedding the field and armature

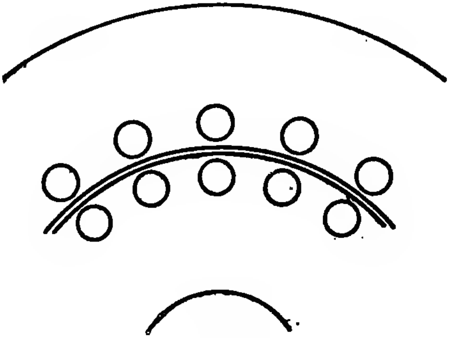

FIG. 109.

windings in the iron instead of having them in the air-gap. The outside conductors of the field winding need not, of course, be embedded. The method frequently adopted in such cases will be understood by reference to Fig. 109, which shows part of the field and armature cores, separated by a narrow air-gap, and perforated with series of holes close to the gap; these holes receive the winding, which is divided into sections. Instead of round holes, rectangular ones (with the corners well rounded) are sometimes used; and not unfrequently the crown which separates the hole or channel from the gap is broken through with a saw-cut.

155. Reference has already been made (Art. 140) to the difficulties which occur in starting polyphase motors. The slip at starting being 100 per cent., we have very large E.M.F.'s induced in the short-circuited coils, and yet the torque of the motor may be comparatively small. In the case of motors of large size, it is found necessary to have a special starting device in order to prevent the abnormally large currents which circulate in the field and armature coils from burning up the insulation. Small motors may, however, be started without fear of any damage to the coils.

Of the various starting devices used, the most common arrangement is to have a simple resistance which may be inserted either into the field or the armature circuit. In the latter case not only is the current in the armature windings kept down, but the torque of the motor is increased at the same time (Art. 140). Special devices must, of course, be used for introducing the resistances into the armature coils at starting and then cutting them out. When the starting resistances are in the field circuit the arrangement is somewhat simpler, but it has the disadvantage of giving a weak starting torque.

156. Polyphase alternators will run as *synchronous* motors, and may be made self-starting, although they

will not start against a heavy load. Let the field of such a polyphase generator be kept unexcited, and let its armature be supplied with polyphase currents. The result is to produce a rotating magnetic field which gives rise to induced currents in the pole-pieces; these currents resist the motion of the rotating field, so that the alternator field magnet tends to follow the motion of the rotating field. If free to move, the field magnet will thus be set in rotation, and run up to a speed which is but little below that of synchronism; the constant exciting current may then be switched on, and the machine will be pulled round into perfect synchronism, and maintained at the constant speed corresponding to it. If, however, the field magnet is stationary, then the armature will begin to rotate in the opposite direction, and synchronism will be reached when the speed of the rotating armature *field* in one direction equals that of the armature *core* in the opposite; it is evident that under these circumstances the field due to the polyphase currents will be a stationary one *in space*, although revolving relatively to the armature. As soon as synchronism is reached, the induced currents in the pole-pieces disappear, the field having become stationary.

This method of starting a synchronous polyphase motor is wasteful; it may be assisted by providing the pole-pieces with short-circuited copper coils. Another and more economical method is to use a small non-synchronous induction motor for running the large motor up to its speed of synchronism.

157. Attempts have been made to adapt rotary-field induction motors for working on ordinary single-

phase alternate-current mains. Before this can be accomplished it is obvious that some method must be provided for producing two currents differing in phase, from the single-phase alternating current supplied by the mains. One way of doing this is to arrange the two field windings of the induction motor so that their time-constants will be widely different, that of one winding being large, and of the other small. If the two windings are connected in parallel across the mains, as in Fig. 110, the currents will lag behind the E.M.F. of the mains by different amounts, the lag being considerable for the winding with a large time-constant, and small for the other. Since the two alternating fields which give rise to the rotating field must be equal in intensity, it follows that the ampere-turns in the two windings of the motor must be made equal. To secure this condition, and at the same time obtain the required difference of phase, one of the windings, W_1 (Fig. 110), consisting of a large number of turns, is connected directly across the mains; while the other, W_2, consisting

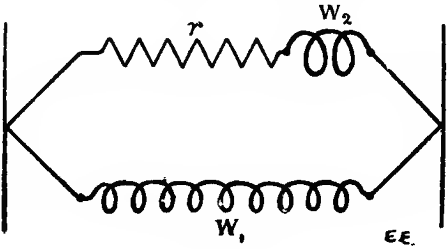

FIG. 110.

of fewer turns (and hence having a much smaller inductance), is joined in series with a non-inductive resistance, r, and across the mains. It is impossible by this method to produce a phase difference of $\frac{\pi}{2}$ (or 90deg.), since the angle of lag of each current behind the E.M.F. must have a value which is intermediate between 0 and $\frac{\pi}{2}$. But a phase difference of about $\frac{\pi}{4}$ (or 45deg.) may be obtained without difficulty. In accordance with the result deduced in Art. 143, the angle which the positive directions of the magnetic fields due to the two coils make with each other must be made equal to the supplement of the angle of phase difference. This is the arrangement adopted in the Langdon-Davies motor.

158. Another class of alternate-current induction motor is that in which the armature rotates in a simple *alternating* field. To understand the operation of such a motor it will be necessary to refer to the curves of Fig. 95 (Art. 140). The full-line curve there given is the typical torque-speed curve of a rotary-field induction motor. It shows that as the *slip* increases the torque at first rapidly increases and then steadily diminishes. This effect has been fully explained in Arts. 138-140. The slip may be made to exceed 100 per cent. by rotating the armature in a direction opposed to that of the field. The torque (which will now resist the motion) will go on decreasing as the negative speed of the armature increases, and we obtain a curve such as the full-line curve shown in

Fig. 111. Let us suppose that the direction of the rotating field is reversed; then instead of the full-line curve of Fig. 111, we obtain the dotted curve, which is merely the full-line curve replotted, the positive direction of the axis of *speed* being now from right to left instead of from left to right as in the first case, and the positive direction of the axis of *slip* from left to right instead of from right to left.

Let us next suppose that there are two rotating fields of the same periodicity and intensity, one revolving right-handedly, the other left-handedly, acting on the armature at the same time.* The torque due to the first field will then be represented by the full-line curve of Fig. 111, and that due to the other field by the dotted curve. For any given speed, positive or negative, the resultant torque (shown chain-dotted in Fig. 111) on the motor armature is equal to the algebraical sum of the torques due to

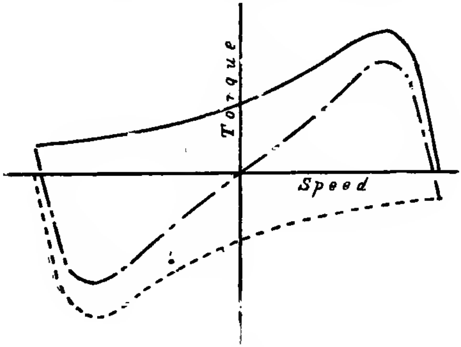

FIG. 111.

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* This arrangement is, by Art. 146, equivalent to a simple alternatin field.

the two rotating fields. Now from the curves it is at once evident that the only speeds for which the torque vanishes are a zero speed, and one which is not far removed from the speed of synchronism. There will thus be no torque acting on the motor when it is standing still. As soon, however, as the motor is turned, either one way or the other, a resultant torque makes its appearance whose sign is the same as that of the speed; the motor will therefore gain speed, and run up to very nearly the speed of synchronism. If a load is applied to it, its speed will diminish, just as in the case of a simple rotary-field motor. And if the resisting torque due to the load is made greater than the maximum torque which the motor is capable of exerting, the armature will stop, and the torque entirely cease.

It is obvious that induction motors of this type are not self-starting, since no torque is produced by the alternating field on the armature when at rest. For starting them, an auxiliary winding must be provided on the field magnet, so arranged that the motor at starting may be temporarily transformed into a rotary-field motor. The requisite difference of phase between the currents in the two windings may be obtained by the method explained in Art. 157.

159. When dealing with polyphase systems, we regarded them as applicable to power plants only. But they are also applicable to lighting; in the latter case, however, difficulties present themselves when the circuits are unequally loaded, on account of the large variations which occur in the P.D.'s between the mains. The regulation of the P.D.'s when the

system is unbalanced must be carefully attended to, and this is the radical difficulty connected with combined power and lighting plants worked on a polyphase system. In some cases, this difficulty appears to have been successfully overcome.

160. In order to secure the advantages of the ordinary single-phase system for lighting along with those of a polyphase system for induction motors of the rotary-field type, Mr. C. P. Steinmetz has devised an arrangement known as the "monocyclic system," which has been used to some extent in America. The principle of this will be understood by reference to Fig. 112. A monocyclic generator is an ordinary alternator provided with a supplementary or "teaser" winding, so placed that the E.M.F.'s in the two windings are in quadrature with each other (Art. 149). In Fig. 112 AB represents the main winding; to the middle point of this, O, is connected one end of the auxiliary winding, O C, the other end of which, C,

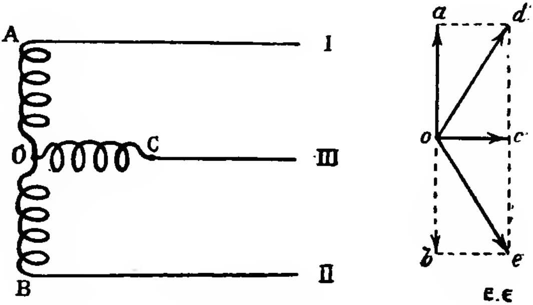

FIG. 112.

is in connection with an auxiliary main intended for supplying induction motors. In order to find the amplitudes and phase relations of the P.D.'s between the mains I., II., and III., we lay off a vector *o a*[*] to represent the P.D. between O and A due to the winding O A, and at right angles to this a vector *o c* to represent the P.D. between O and C due to O C; the resultant vector *o d* then represents the P.D. between the mains I. and III.; similarly, *o e* represents the P.D. between II. and III., while *b a* or *e d* gives the P.D. between I. and II. It is obvious that by properly adjusting the number of turns in the auxiliary winding O C, and hence the length of the vector *o c*, we can make the P.D.'s represented by *o d* and *o c* bear any desired relation to *e d* as regards amplitude and phase. The mains I. and II. would be used for lighting, while polyphase motors would be connected to I., II., and III. The system therefore combines the characteristics of ordinary single-phase working with the advantages of a polyphase system for induction motors.

Notes on Chapter XIX.

One of the distinguishing features of all polyphase systems as compared with a single-phase system lies in their extreme flexibility. With a polyphase system, the transition from alternating to continuous current is accomplished quite readily. Mr. Steinmetz's monocyclic system is intended to supply a connecting link between single- and multi-phase working. A comparison of the two-phase with the three-phase system shows that the former is, generally speaking, more convenient for distribution purposes, while the latter is better for transmission. In order to combine the advantages of both systems, Mr. C. F. Scott has patented in America a method of transforming two-phase into three-phase currents, and *vice versâ*. A description of the Scott system will be found in the *Electrical World* of New York, Vol. XXIII., p. 393.

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[*] All the vectors here referred to are supposed as usual to denote rotating vectors whose projections on the vertical axis give the corresponding values of the P.D.'s.

CHAPTER XX.

161. The subject of power measurement on ordinary single-phase circuits has been dealt with in Arts. 69-72, and it was there stated that the most generally useful and reliable method was that involving the use of a non-inductive wattmeter. This latter instrument is equally useful in the case of polyphase circuits; there are, however, certain points connected with such circuits which render the measurement of power a somewhat more complicated process than in the case of ordinary single-phase circuits.

162. We shall begin by considering a two-phase (or di-phase) system. If the two circuits of such a system are completely independent of each other the power may be measured separately in each circuit, the connections being made as in Fig. 54. The sum of the two readings gives the total power of the two-phase circuit.

If the circuits have one wire in common, as in Fig. 104, the same method may still be used. Two wattmeters are connected as shown in Fig. 113, the current coils, C_1 and C_2, of the wattmeters being placed in the *outside* wires, while their fine-wire coils are connected across the outside mains on the one hand, and the central or common main on the other. The sum of the two wattmeter readings gives the total power.

If the load on a two-phase circuit consists partly of motors and partly of lamps, then, as a rule, the distribution of load between the two circuits will be unequal, and we must use two wattmeters (or take two consecutive readings as rapidly as possible with the same wattmeter when placed in each of the two circuits) as explained above. But if the plant be entirely a power plant with two-phase motors, the load is necessarily the same in each circuit. In such a case the system is said to be a *balanced* one. It is obvious that a single wattmeter reading will in that case suffice, the total power being equal to twice the reading of the wattmeter.

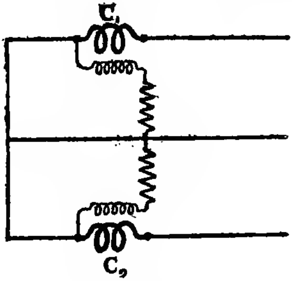

Fig. 113.

163. Passing next to three-phase systems, we shall in the first place take the general case in which the load on each of the three circuits is different.

Consider a three-phase system with a *star* coupling, as shown in Fig. 108. If the neutral point (Art. 153) is accessible, the simplest plan is to use three wattmeters, connecting the current coils in series with three circuits, and the fine-wire coils across the neutral point and the three terminals respectively. This arrangement of connections is shown in

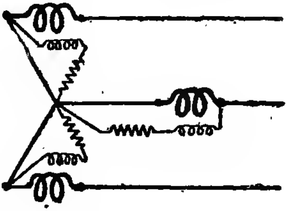

FIG. 114.

Fig. 114. The sum of the three wattmeter readings then gives the total power.

If the circuits are arranged in the form of a *mesh* or triangle (Art. 152), as in Fig. 106, and if it be possible to introduce a current coil into each of the three

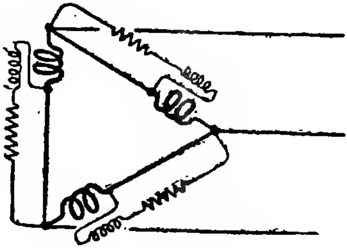

FIG. 115.

circuits, as shown in Fig. 115, the fine-wire coils being connected between the terminals, then the sum of the three wattmeter readings will as before give the total power.

The conditions required by these methods are seldom if ever met with in practice. The neutral point is not generally accessible, and it is not as a rule possible to connect up the wattmeters as in Fig. 115. Hence some other methods for measuring the power in a three-phase circuit must be devised.

164. One of these consists in the creation of an artificial "neutral point" by the use of three equal non-inductive resistances connected as in Fig. 116. The power may then be obtained by taking the sum of the readings of three wattmeters, the current coils of which are in series with the three conductors of the system, while the fine-wire coils, with their non-inductive resistances, are joined across the artificial neutral point and the three mains respectively. The arrangement of connections is clearly shown in Fig. 116, where r_1, r_2, and r_3 denote the three equal non-inductive resistances, N being the artificial neutral point, C_1, C_2, and C_3 the thick-wire coils of the three wattmeters, and f_1, f_2, and f_3 their fine-wire coils. The method is

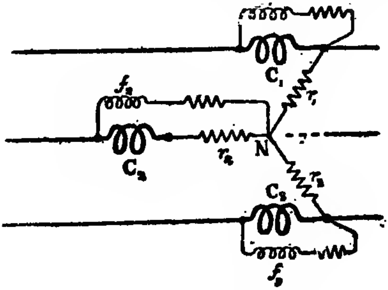

Fig. 116.

obviously applicable to either a "star" or "mesh" arrangement of circuits. If the fine-wire circuits of the three wattmeters have the same resistance, the three non-inductive resistances, r_1, r_2, r_3, may be dispensed with, the neutral point being in this case formed by the junction of the fine-wire circuits themselves.

165. Another method involves the use of two wattmeters only. The arrangement of connections is indicated in Figs. 117 and 118. The thick-wire coils of the two wattmeters are placed in two of the mains, and their fine-wire coils are joined across these two mains respectively and the third main. We proceed to show that whether the arrangement of the circuits be a "star" or a "mesh" combination, the total power may be determined from the readings of the two wattmeters.

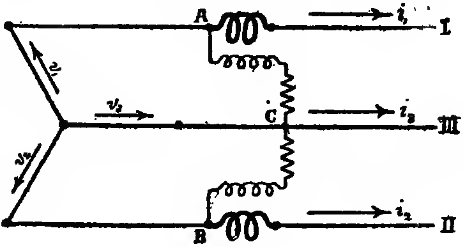

FIG. 117.

Consider first a star grouping, as shown in Fig. 117. Let i_1, i_2, and i_3 denote the instantaneous currents in the three mains, reckoned positive when flowing from left to right; and let v_1, v_2, and v_3 stand for the instantaneous P.D.'s between the neutral point and the three mains respectively, reckoned positive when acting *away from* the neutral point. The instantaneous power is then given by

$$w = v_1 i_1 + v_2 i_2 + v_3 i_3.$$

But we also have

$$i_1 + i_2 + i_3 = 0,$$

whence $$i_3 = - i_1 - i_2.$$

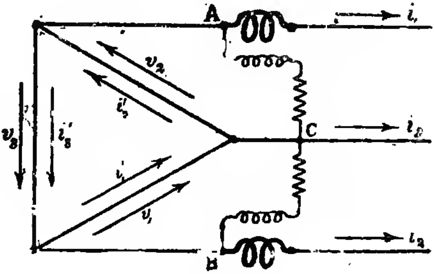

FIG. 118.

Substituting this value of i_3 in the expression for the power, we get

$$w = (v_1 - v_3)\, i_1 + (v_2 - v_3)\, i_2.$$

Let the two wattmeters shown in Fig. 117 be so connected that when the currents through their main coils have the directions indicated by the arrows, and when the P.D.'s acting on their shunt coils have directions from A and B *towards* C respectively, the corresponding couples on the watt-meters are positive. Under these circumstances it is evident that the wattmeter whose main coil is in I. measures the mean value of the product $(v_1 - v_3)\,i_1$, and the wattmeter whose main coil is in II. the mean value of $(v_2 - v_3)\,i_2$. From this it follows that the *algebraical sum* of the readings of the two wattmeters gives the mean value of w—*i.e.*, the mean power in the three-phase circuit.

166. Let us next suppose that the three circuits are arranged in a mesh, as in Fig. 118, the wattmeter connections remaining unaltered. Let e_1, e_2, and e_3, stand for the instantaneous E.M.F.'s in the three alternator circuits; v_1, v_2, and v_3 for the corresponding P.D.'s between the mains; and i_1', i_2', i_3', for the currents. All these quantities are reckoned positive when acting in a counter-clockwise direction round the mesh, as indicated by the arrows. On account of the symmetry of the three circuits in the polyphase generator, the resistance, r, of each circuit will be the same, and we have

$$e_1 = v_1 + r\,i_1'$$

$$e_2 = v_2 + r\,i_2'$$

$$e_3 = v_3 + r\,i_3',$$

whence $$e_1 + e_2 + e_3 = v_1 + v_2 + v_3 + r\,(i_1' + i_2' + i_3').$$

But $$e_1 + e_2 + e_3 = 0,$$

and $$i_1' + i_2' + i_3' = 0,\text{*}$$

so that $$v_1 + v_2 + v_3 = 0.$$

---

* Since there is no local current round the mesh.

The instantaneous power is given by

$$\begin{aligned} w &= v_1 i_1' + v_2 i_2' + v_3 i_3', \\ &= v_1 i_1' + v_2 i_2' - (v_1 + v_2) i_3', \\ &= v_1 (i_1' - i_3') + v_2 (i_2' - i_3'), \\ &= -v_1 i_2 + v_2 i_1, \end{aligned}$$

from which it will be seen that the algebraical sum of the readings of the two wattmeters gives the total power in the three-phase circuit, provided the connections are made (as before) so that when the currents in the two mains containing the thick-wire coils are from left to right, and the P.D.'s on the fine-wire coils are directed *towards* C, the resultant couples on the two wattmeters are both positive.

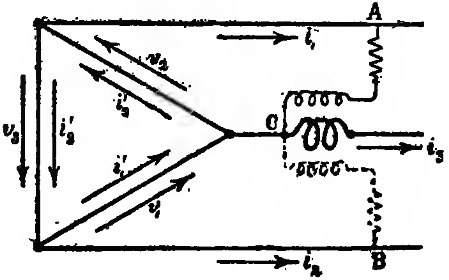

FIG. 119.

167. In the case of a *balanced* three-phase system, a single wattmeter is sufficient for measuring the total power. The current coil of the wattmeter is placed in one of the mains (Fig. 119), and one end of the fine-wire coil is also connected to the same

main. Two readings are taken, with the other end of the fine-wire coil connected to each of the remaining two mains respectively. The sum of the two readings gives the total power. The proof of this result is similar to that used in Art. 166. Assuming a mesh arrangement of the circuits, as in Fig. 119, and using the result already established in Art. 166, we have for the instantaneous power in the three-phase circuit

$$\begin{aligned} w &= -v_1 i_2 + v_2 i_1 \\ &= v_1 i_1 + v_1 i_3 - v_2 i_2 - v_2 i_3. \end{aligned}$$

Now, since the circuit is a *balanced* one, it follows from its symmetry (Fig. 119) that the mean value of $v_1 i_1$ is the same as that of $v_2 i_2$. The expression for the instantaneous power now becomes

$$w = v_1 i_3 - v_2 i_3,$$

from which it follows that if the thick-wire coil of a wattmeter be introduced at C (Fig. 119), one end of the fine-wire coil being connected to the same point, and two readings be taken, one with the free end of the fine-wire coil connected to A, and the other to B, the sum of the two readings will give the total power in the balanced circuit.

A similar method of proof shows that the measurement is equally applicable to a star grouping.

ANSWERS.

I.

2. 11; 38. **3.** 7; 11; 15; 19. **4.** 3·87; 4·2; 3·4; 2·8; 2·36.

II.

1. 42; 17°. **2.** (*a*) 104·7 radians per sec.; (*b*) 43·6ft. per sec. **3.** (*a*) $\frac{\pi}{3}$ radians, or 60°; (*b*) ·0006 second. **4.** 9,893,000.

III.

1. ·2 volt. **2.** 120 volts. **3.** 2 volts. **4.** ·106 ft.-lb.

IV.

2. ·002 sec. **3.** ·0161 sec. **4.** ·00322 sec. **5.** 4·88 amperes; 78° 45′. **6.** ·0317 henry.

V.

3. $10 \sin p\,t - 3 \sin 3\,p\,t$.

VII.

1. 4·48 amperes. **2.** 1·18.

VIII.

1. 92 watts. **2.** ·856. **3.** 11·3 volts; by 28·4 %. **4.** 17·8 watts. **5.** ·0101 henry.

XI.

1. ·033 ft.-lb. **2.** 167,000 ergs. **3.** ·064 ampere.

XII.

1. 1,052 volts. **2.** 1·3 ampere. **3.** 4·46 amperes; 60·3 volts and 355 volts respectively. **4.** 7·23 microfarads. **5.** 4·23 microfarads.

XIII.

1. 4,000 microcoulombs. **2.** ·00101 henry. **3.** 360,000 ergs.

XIV.

1. 18·85 volts. **2.** ·653 ampere. **3.** 6·23 ampere.

XVI.

1. ·486 volt (coil and field rotating in same direction). **2.** ·811 ampere ; ·0923 pound-foot.

XVII.

1. 135°.

XVIII.

1. Amount of copper in case (*a*) is to that in case (*b*) as 8 is to $3 + 2\sqrt{2}$, or as 100 : 73 approximately. **2.** (*a*) £9,135 ; (*b*) £9,105 ; (*c*) £3,037. **3.** Amount of copper in case (*a*) is to that in case (*b*) as 100 : 145.

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